

Potts model of magnetism (invited)

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The Potts model is a generalization of the Ising model of magnetism to more-than-two components. First considered by Potts in 1952, the problem has aroused considerable interest in recent years. It has been shown that the model is very rich in its content and, in addition, the extra degree of freedom exhibited by the number of components permits the model to be realized in a wide range of physical systems. In this paper we review those aspects of the Potts model related to its contents as a model of magnetism, focusing particular attention to the results obtained since a previous review was written. Topics reviewed include the upper and lower critical dimensionalities, critical properties, and some exact and rigorous results, for both the ferromagnetic and antiferromagnetic models.

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I. INTRODUCTION

The Potts model, introduced by Potts¹ more than thirty years ago as a generalization of the Ising model of magnetism, has attracted increasing recent attention. It is now known that the critical behavior of the Potts model is very rich and more general than that of the Ising model, and that the model can be realized in many different physical systems. It is also known that the Potts model is related to a number of outstanding statistical problems. Discussions of these and other related topics of the Potts model have been given in a recent review² which summarizes the status of our understanding of the problem as of 1981. A large quantity of new results on the Potts model have since been accumulated, and it would not be possible to discuss all these developments without engaging a major endeavor. The purpose of this paper is to present a self-contained, albeit limited, review of the Potts model in its role as a model of magnetism, paying particular attention to those developments since the previous review was written. Due to the limitation of space, however, derivations of the results will not be given; a number of other pertinent developments, such as those related to the chiral Potts models not directly related to magnetism, will also be omitted.

Consider a system of N spins located on a lattice such that each spin can have q values (states) described by, say, $\sigma = 1, 2, \dots, q$. The nearest-neighbor spins have an interaction energy $-\epsilon$ if their spin values are alike, zero if they are different. Thus the partition function is

$$Z(q, K) = \sum \exp \left[K \sum_{nn} \delta(\sigma, \sigma') \right], \quad (1)$$

where $K = \epsilon/kT$. Here the first summation is over all spin configurations, and the summation in the exponential is over all nearest-neighbor pairs of the lattice.

The system is ferromagnetic if $K > 0$. In this case the ground state consists of configurations in which all spins are in the same state. Furthermore, the system will exhibit, at sufficiently low temperatures, a spontaneous magnetization showing an ordering in one of the q spin states. For $K < 0$ the system is antiferromagnetic for which the ground state is one in which two nearest neighbors have distinct spin values. Thus, from the viewpoint of the ground-state orderings (and disorderings), the Potts model generalizes the two-component Ising model to q components.

The thermodynamics of the Potts model are derived from the "free energy" defined by taking the thermodynamic limit of the logarithmic partition function

$$f(q, K) = \lim_{N \rightarrow \infty} N^{-1} \ln Z(q, K). \quad (2)$$

From Eq. (2) one obtains the energy and specific heat, respectively,

$$E(q, K) = -\epsilon \frac{\partial}{\partial K} f(q, K), \quad (3)$$

$$C(q, K) = kK^2 \frac{\partial^2}{\partial K^2} f(q, K), \quad (4)$$

which, in turn, determine the nature of the transition. The spontaneous magnetization is defined by²

$$M = [q \langle \delta(\sigma, 1) \rangle - 1] / (q - 1), \quad (5)$$

where σ is a spin located in the interior of the lattice, and $\langle \rangle$ denotes the thermal average taken with all spins at the boundary fixed in the spin state 1. The spontaneous magnetization M vanishes identically for $T > T_c$, where T_c is the critical temperature, and its exact expression for $T < T_c$ is known for the two-dimensional $q = 2$ (Ising) model only.

While the parameter q enters Eq. (1) as an integer, one can always analytically continue the expressions (2)–(5) to arbitrary, even complex, values of q . This can be accomplished by continuing q , for example, after the summations in the partition function have been taken so that q appears in

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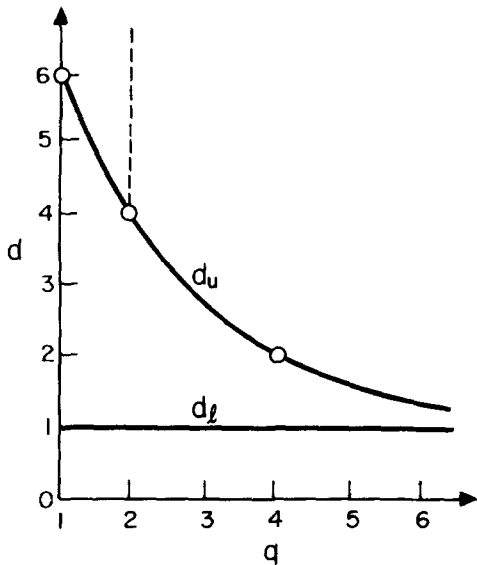


FIG. 1. Upper and lower critical dimensions of the ferromagnetic Potts model. The circles denote the exactly known points.

$Z(q, K)$ as a parameter, rather than a summation index. This procedure considerably generalizes the Potts model (1).

II. CRITICAL DIMENSIONS

It is illuminating and often useful to regard the thermodynamic properties of a system as functions of the dimensionality d of the underlying lattice. One then defines the critical dimensions as the values of d at which the critical behavior experiences a change or simplification. This critical dimension can, in principle, take on integral as well as nonintegral values.

There exists in general two critical dimensions in systems which exhibit some kind of critical behavior: A lower critical dimension d_c characterized by the fact that the system no longer goes critical whenever $d \leq d_c$, and an upper critical dimension d_u characterized by the fact that the critical behavior suffers a change, such as becoming mean-field-like, for $d \geq d_u$. For the Potts model we would generally expect, unless otherwise indicated, both d_c and d_u to be q dependent.

Consider first the ferromagnetic case. Since it is exactly known that the ferromagnetic model exhibits a transition in two dimensions,³ for $q > 1$ at least, and no transition in one dimension, it is not unreasonable to regard, as is verified by the renormalization group result for $q = 2$,⁴ that the lower critical dimension to be one for all q . This constant $d_c = 1$ is shown by the horizontal line in Fig. 1.

The upper critical dimension for the ferromagnetic model is the dimension beyond which the system becomes mean-field-like, or the fluctuation becomes unimportant. Now the mean-field solution leads to a phase transition which is continuous for $q \leq 2$ and first order for $q > 2$.² Thus $q = 2$ is a border in the mean-field regime and is depicted by the broken line segment in Fig. 1. Also shown in Fig. 1 is a (schematic) plot of the upper critical dimension $d_u(q)$ passing through three known exactly points, $(q, d) = (1, 6), (2, 4), (4, 2)$.² It should be noted that the transition is always continuous for $1 < q < 2$ and $d > 1$, with the upper critical dimen-

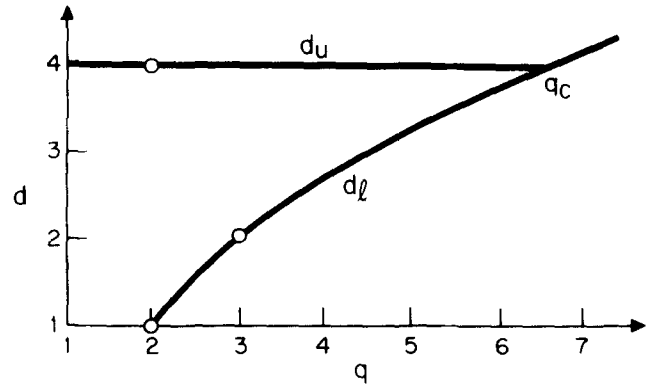


FIG. 2. Upper and lower critical dimensions of the antiferromagnetic Potts model. The circles denote the exactly known points. The two critical dimensionalities coincide for $q \geq q_c$.

sion separating the classical (cusp singularity in the specific heat) and non-classical (divergent specific heat) regions.

Consider next the case of the antiferromagnetic model. Berker and Kadanoff⁵ have argued on the basis of a rescaling consideration that a q -dependent lower critical dimension $d_c(q)$ should exist, and obtained its numerical estimates. Phenomenological⁶ and Monte Carlo⁷ renormalization group studies in two dimensions and subsequently an exact analysis for the square lattice⁸ have established the exact result that $d_c(3) = 2$ (see Sec. IV below). In addition, it has been further established that $d_c(2) = 1$.⁹ Thus, we now have two exact points, $(q, d) = (2, 1), (3, 2)$ for the lower critical dimension. It is also expected the lower critical dimension to behave as,¹⁰ for large q ,

$$d_c(q) \sim \ln q, \quad q \rightarrow \infty. \quad (6)$$

A (schematic) plot of $d_c(q)$ reflecting these behaviors is shown in Fig. 2.

Consider now the upper critical dimension $d_u(q)$. We again regard it to be the dimension beyond which the system becomes mean-field-like. Now the mean-field solution of the antiferromagnetic model leads to a continuous transition for all q .¹¹ Further, for bipartite lattices, the $q = 2$ ferromagnetic and antiferromagnetic models are isomorphic. These considerations lead to an exact point $d_u(2) = 4$.

For other values of q we use the fact that, if a transition exists and if the transition is continuous in the antiferromagnetic Potts model, then it is in the same universality class of the $O(n)$ model with $n = q - 1$.^{12,13} Now the upper critical dimension of the $O(n)$ model is 4 for all $n(\geq 1)$.¹⁴ It is then very plausible that, as q increases from 2 (and n increases from 1), the upper critical dimension of the q -state antiferromagnetic Potts model remains to be 4 until q reaches a critical value q_c defined by

$$d_c(q) = 4. \quad (7)$$

Beyond q_c , the upper and lower critical dimensions coalesce and the transition is always classical (mean-field-like). This conjectured behavior of the upper critical dimension for bipartite lattices is shown in Fig. 2.

III. CRITICAL PROPERTIES

The ferromagnetic Potts model exhibits a phase transition in two dimensions, and that the transition is continuous

with nonclassical exponents for $q \leq 4$, and is first order, i.e., accompanied with a latent heat, for $q > 4$. A summary of these and other relevant critical properties, including a list of critical exponents and expressions for critical quantities can be found in Ref. 2. More recently using the corner transfer matrix approach, Baxter¹⁵ has shown that the spontaneous magnetization M is discontinuous at the critical temperature T_c for $q > 4$, jumping from the value zero for $T > T_c$ to a nonzero value $M(T_c^-)$ at T_c . The exact expression for this jump discontinuity obtained by Baxter is

$$M(T_c^-) = 0 \quad q \leq 4$$

$$= \prod_{n=1}^{\infty} [(1 - e^{-(2n-1)\theta}) / (1 + e^{-2n\theta})] \quad q > 4, \quad (8)$$

where

$$q = 2 \cosh \theta. \quad (9)$$

It is remarkable that the expression (8), which depends only on q , not on the interactions K , is valid for all two-dimensional lattices,¹⁵ a fact first conjectured by Kim.¹⁶ The expression (8) possesses an essential singularity at $q = 4$ near and above which it behaves as

$$M(T_c^-) \sim 2 \exp[-\pi^2/8(q-4)^{1/2}]. \quad (10)$$

This is in agreement with the renormalization group prediction.¹⁷

A first-order transition exists in the Potts model in $d \geq 2$ dimensions when q is sufficiently large. This fact, while expected intuitively on the basis of the mean-field analysis, has recently been rigorously proved.¹⁸ One can also use this fact to establish the existence of a first-order transition in certain Ising models which are equivalent to a Potts model of $q = 2^n$, $n = 1, 2, \dots$, components.¹⁹

The critical exponents of the Potts model are well defined when the transition is continuous. To obtain the two leading thermal exponents we expand the singular part of the free energy (2) about the critical point K_c :

$$f_{\text{sing}}(q, K) \sim |K - K_c|^{2-\alpha} [1 + a_1 |K - K_c|^{\Delta_1} + \dots]. \quad (11)$$

The thermal exponents are obtained from Eq. (11) using the relations $y_1 = d/(2-\alpha)$ and $\Delta_1 = -y_2/y_1$. In a similar fashion for the two leading magnetic exponents we write, for r large, the critical ($K = K_c$) two-point correlation function $\Gamma_c(r)$ as²⁰

$$\Gamma_c(r) \sim r^{-(d-2+\eta)} [1 + a_2 r^{-\Delta_2} + \dots], \quad r \rightarrow \infty. \quad (12)$$

The exact value of the leading thermal exponent for $d = 2$ is²¹

$$\alpha = 2(1 - 2u)/3(1 - u), \quad (13)$$

where

$$0 \leq u = (2/\pi) \cos^{-1}(\sqrt{q}/2) \leq 1, \quad 0 \leq q \leq 4. \quad (14)$$

Nienhuis²² has further computed the correction to scaling and obtained

$$\Delta_1 = 4u/3(1 - u). \quad (15)$$

The exact values of the two leading magnetic exponents for the two-dimensional models have been derived by den Nijs²⁰

from a consideration of the Coulomb-gas representation of the Potts model. The results are

$$\eta = (1 - u^2)/2(2 - u), \quad (16)$$

$$\Delta_2 = 4/(2 - u). \quad (17)$$

Thus, $\Delta_1 = 4, 8/3, 4/3, 2/3, 0, \Delta_2 = 1, 3, 8/3, 12/5, 2$, for $q = 0, 1, 2, 3, 4$ respectively. The $q = 3$ value of $\Delta_1 = 2/3$ is to be compared with the numerical estimates of $\Delta_1 \sim 0.6$ from a low-temperature series analysis.²³

The partition function of the Potts model satisfies an inversion functional relation which is most easily derived by considering its transfer matrix.^{24,25} For the square lattice with anisotropic interactions K_1 and K_2 , the inversion relation reads

$$Z(q, K_1, K_2) Z[q, -K_1, \ln(2 - q - e^{K_2})] = (e^{K_1} - 1)(1 - q - e^{-K_2}), \quad (18)$$

where $Z(q, K_1, K_2)$ is the corresponding partition function of the anisotropic model. The validity of Eq. (18) has been verified by perturbative large- q expansions.^{24,26} The inversion relation (18), which is based on the transfer matrix formalism, can be readily generalized to other, including the checkerboard and the simple cubic, lattices.^{27,28} However, unlike the $q = 2$ Ising case for which the inversion relation can be used in conjunction with an analytic assumption to uniquely derive the partition function,²⁹ the inversion relation for general q does not seem to lead to any determination of the Potts partition function. There also appears to be some profound differences between the inversion relations of the $d = 2$ and $d = 3$ models.²⁸

The critical point of the Potts model are exactly known for the square, triangular, and honeycomb lattices.² But using the inversion relation one can further locate the critical point for the checkerboard lattice with interactions K_1, K_2, K_3 , and K_4 . This leads to the following critical manifold²⁷:

$$\prod_{i=1}^4 [(e^{K_i} + e^{2\theta}) / (e^{K_i + \theta} + e^{-\theta})] = 1. \quad (19)$$

The expression (19) confirms an earlier conjecture.³⁰ It is of some interest to note that Svrakic³¹ used a heuristic argument to deduce the exact critical point for some two-dimensional models, but the argument does not reproduce Eq. (19) when applied to the checkerboard lattice.

For three-dimensional lattices Park and Kim³² have obtained accurate estimates of the critical point from an analysis of the large- q series expansions of the susceptibility and magnetization. More generally, Hajdukovic³³ proposed an expression for the critical point of the q -state model on a d -dimensional hypercubic lattice. His conjecture,

$$e^{dK} - 2^{1/(d-1)} e^{dK/2} - q + 1 = 0, \quad (20)$$

is indeed exact for $d = 2$ and $d = 1$, and is fairly accurate for $q \geq 2$ in $d \geq 3$. But it gives an erroneous critical probability of $p_c = 1 - 2^{-1/3} = 0.2063$ for the bond percolation ($q = 1$) on the simple cubic lattice ($d = 3$). Since this prediction differs appreciably from the value $p_c = 0.247$ derived from accurate numerical analyses,³⁴ the expression (20) cannot be correct for general q and d .

IV. ANTIFERROMAGNETIC MODEL

The antiferromagnetic Potts model has received considerable attention in the last two years. It has been studied under different approaches including the finite-size⁶ and Monte Carlo⁷ renormalization groups, phenomenological scaling transformation,³⁵ mean-field analysis,¹¹ Monte Carlo simulations,^{11,12,36} and exact analysis.⁸ Particularly, Baxter⁸ observed that the two-dimensional antiferromagnetic model (1) is exactly solvable at

$$(e^K + 1)^2 = 4 - q \quad (21)$$

and argued that, as in the case of the ferromagnetic model, the solubility condition (21) should coincide with the critical point and that the transition at Eq. (21) is continuous. We see from Eq. (21) that the transition occurs at real temperatures only for $q < 3$. The vanishing of the transition temperature at $q = 3$ establishes the exact result of $q_c(3) = 2$ quoted in Sec. II.

Most of the known results of the Potts model are derived from its equivalence with a six-vertex model³⁷ and the established results of the latter problem. However, in order to establish the equivalence of the two models, it is necessary to assume a boundary condition for the Potts model which is periodic in no-more-than-one direction. On the other hand, the known results of the six-vertex model are always derived by assuming periodicity in two directions. Therefore, care must be taken in transcribing the results. Indeed, one can even see in one dimension that the antiferromagnetic Potts model is very sensitive to the boundary conditions. The introduction of a periodic boundary condition to a one-dimensional chain, e.g., induces a transition for $1 < q < 2$.⁹ In two dimensions, Baxter⁸ observed that the solution of the six-vertex model with periodic boundary conditions does *not* always yield the correct critical free energy for the antiferromagnetic Potts model with free boundaries. While this particular difficulty is circumvented by the use of the inversion relation, there is a clear need for alternate approaches to the antiferromagnetic problem.

In the absence of further exact data, the antiferromagnetic model is best studied by Monte Carlo simulations.^{11,12} It has been established that, for bipartite lattices and except for $q = 3$ and $d = 2$ for which there is no transition, a low-temperature ordered phase exists with distinct spin states favored on each of the two sublattice. This is the antiferromagnetic (AF) phase, and the transition into which is found to be continuous. However, the critical properties associated with this transition remain undetermined. In this connection, the universality-class equivalence of the q -state antiferromagnetic model and the $O(q - 1)$ model (cf. Sec. II) should prove useful in further studies. It should also be borne in mind that the lattice symmetry, whether bipartite or tripartite, also plays a crucial role in determining the critical properties.

The picture is more complicated when there are next-nearest-neighbor competing ferromagnetic interactions. Finite-size scaling analysis of the 3-state model has led to some understanding of the phase diagram as well as numerical estimates of the critical exponents in two dimension.³⁸ It also suggests that the model is in the same universality class

of the 6-state clock (planar Potts) model.³⁸ In three dimensions, Monte Carlo study¹¹ of the 4-state model indicates a mean-field-like phase diagram. That is, for small ferromagnetic interactions the system exhibits a sequence of three transitions. At low temperatures the system possesses a net "ferromagnetic" ordering. As the temperature rises, the system first changes the ordering to one characterized by a broken sublattice symmetry (BSS) for which one spin state is favored on one sublattice and the other $q - 1$ states on the other sublattice. The system next goes into the AF phase, and eventually becoming paramagnetic (P) at high temperatures. For larger ferromagnetic interactions, the AF and BSS phases disappear in succession, leaving finally a single transition between the ferromagnetic and paramagnetic phases. The mean-field theory predicts that all, except the AF-P, transitions are first order. This prediction appears to be already realized in the $q = 4$, $d = 3$ model.¹¹ But the critical exponents associated with the continuous AF-P transition remain undetermined.

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