

LETTER TO THE EDITOR

Algebraic invariants of the $O(2)$ gauge transformation

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Abstract. We consider the $O(2)$ gauge transformation for a two-state vertex model on a lattice, and derive its fundamental algebraic invariants, the minimal set of homogeneous polynomials of the vertex weights which are invariant under $O(2)$ transformations. Explicit expressions of the fundamental invariants are given for symmetric vertex models on lattices with coordination number $p = 2, 3, 4, 5, 6$, generalising $p = 3$ results obtained previously from more elaborate considerations.

In a study of the symmetry properties of discrete spin systems, Wegner [1] introduced a gauge transformation generalising the weak-graph transformation used by earlier investigators [2-4]. The gauge transformation, which describes important symmetry properties including the usual duality relation [3], is a linear transformation of the weights of a vertex model under which the partition function remains invariant. One particular symmetry property studied for over a century [5] is the construction of algebraic invariants, the homogeneous polynomials invariant under linear transformations. The problem of constructing invariants for the gauge transformation in vertex models has been studied by Hijmans *et al* [6,7] for the square lattice and, more recently, by Wu *et al* [8] and by Gwa [9] for the $O(2)$ transformation on trivalent lattices. Specifically, Wu *et al* [8] proposed that the critical frontier of the Ising model in a non-zero magnetic field is given by the algebraic invariants of the related vertex model, and constructed the invariants by enumeration for trivalent lattices. A simpler method leading to the same invariants was later given by Gwa [9]. But the extension of both of these analyses to lattices of general coordination number p has proven to be extremely tedious, becoming almost intractable for $p > 4$. Clearly, an alternative and simpler approach is needed.

In this letter we consider the $O(2)$ gauge transformation for a two-state vertex model on a lattice of general coordination number p , and present a formulation which leads to a simple and direct determination of its algebraic invariants.

We first define the vertex model and the $O(2)$ gauge transformation. Consider a lattice of coordination number p , with the lattice edges in one of two distinct states independently at each edge. We may regard the edges as being either 'empty' or 'covered' by a bond, so that the edge configurations generate bond graphs [10]. Introduce edge variables $s = 0, 1$ so that $s = 0$ ($s = 1$) denotes the edge being empty (covered). With each lattice site associate a vertex weight $W(s_1, s_2, \dots, s_p)$, where s_1, s_2, \dots, s_p indicate the states of the p incident edges. The partition function of this two-state vertex model is

$$Z(\{W\}) = \sum \prod_i W(s_1, s_2, \dots, s_p) \quad (1)$$

where the summation is taken over all bond graphs of the lattice, and the product is taken over all vertices i .

Consider a linear transformation of the 2^p vertex weights $W(s_1, s_2, \dots, s_p)$,

$$\tilde{W}(t_1, t_2, \dots, t_p) = \sum_{s_1=0}^1 \sum_{s_2=0}^1 \dots \sum_{s_p=0}^1 R_{t_1 s_1} R_{t_2 s_2} \dots R_{t_p s_p} W(s_1, s_2, \dots, s_p). \tag{2}$$

The transformation (2) leaves the partition function invariant if R_{is} are elements of a 2×2 matrix \mathbf{R} satisfying $\tilde{\mathbf{R}}\mathbf{R} = I$, where I is the identity matrix [1]. This implies $\det|R_{is}| = \pm 1$, and therefore the transformation (2) provides a representation of the two-dimensional orthogonal group $O(2)$, to be referred to as the $O(2)$ gauge transformation.

The $O(2)$ group is generated by a rotation $\mathbf{R}^{(1)}$ or a reflection $\mathbf{R}^{(2)}$ given by

$$\mathbf{R}^{(1)} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad \mathbf{R}^{(2)} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}. \tag{3}$$

Note that $\mathbf{R}^{(2)}$ has been used exclusively in previous investigations [2,4,8].

For *symmetric* vertex models, the vertex weights depend only on the number of covered incident edges, for which we have

$$W(s_1, s_2, \dots, s_p) = W_p(s) \quad s = s_1 + s_2 + \dots + s_p \tag{4}$$

where $s = 0, 1, 2, \dots, p$ is the number of bonds incident at the vertex. We shall, however, continue to assume general vertex weights, and only below specialise the results to symmetric vertex models.

Hilbert [5, see also p 235 of Gurevich in [5]] established more than a century ago that invariants of a linear transformation are in the form of homogeneous polynomials, and that all such polynomials are expressible in terms of a minimal set of *fundamental* ones. The crux of the matter is, of course, the determination of these fundamental invariants for a given linear transformation. For the $O(2)$ transformation, as we now show, the task can be accomplished as follows.

Introduce the change of basis

$$A_{\sigma_1 \dots \sigma_p} = \sum_{t_1=0}^1 \dots \sum_{t_p=0}^1 \left[\prod_{k=1}^p (i\sigma_k)^{t_k} \right] W(t_1, \dots, t_p) \tag{5}$$

where $\sigma_k = \pm 1$. For example, for $p = 2$, (5) is

$$\begin{aligned} A_{\pm\pm} &= [W(00) - W(11)] \pm i[W(01)] + W(10) \\ A_{\pm\mp} &= [W(00) + W(11)] \mp i[W(01) - W(10)]. \end{aligned} \tag{6}$$

In a similar fashion we define $\tilde{A}_{\sigma_1 \dots \sigma_p}$ in terms of $\tilde{W}(s_1, \dots, s_p)$. Then, using the identity

$$\sum_{t=0}^1 (i\sigma)^t R_{ts}^{(l)} = (-1)^{s(l-1)} e^{i\sigma\theta} (i\sigma)^s \quad l = 1, 2 \tag{7}$$

where $R_{ts}^{(l)}$ are elements of $\mathbf{R}^{(1)}$ or $\mathbf{R}^{(2)}$ given by (3), one obtains from (5) and (2) the following transformation property for the A :

$$\tilde{A}_{\sigma_1 \dots \sigma_p}^{(l)} = \begin{cases} e^{i(\sigma_1 + \dots + \sigma_p)\theta} A_{\sigma_1 \dots \sigma_p} & l = 1 \\ e^{i(\sigma_1 + \dots + \sigma_p)\theta} \theta A_{\sigma_1 \dots \sigma_p}^* & l = 2 \end{cases} \tag{8}$$

where $\tilde{A}^{(l)}$ is the \tilde{A} obtained from (5) by using \tilde{W} in (2) with $\mathbf{R} = \mathbf{R}^{(l)}$, $l = 1, 2$, and A^* is the complex conjugate of A . For $l = 1$, the A s change only by a phase factor under the $O(2)$ transformation. It follows that any product of the A s, for which the σ s of all A factors in the product add to zero, is an invariant. For $l = 2$, however, the products are transformed into complex conjugates, in addition to the change of a phase factor. Thus, in both cases the real parts of these products are invariant, and the imaginary parts are invariant under $\mathbf{R}^{(1)}$ while changing sign under $\mathbf{R}^{(2)}$. We shall, however, refer to both the real and imaginary parts as the invariants.

For a given p , the fundamental invariants can now be constructed by following the above prescription. For $p = 2$, there are three fundamental invariants $A_{++}A_{--}$, A_{+-} , and A_{-+} , the last two being the complex conjugates of each other; any other invariant is composed of the three. For $p = 3$, our consideration leads to 30 distinct products, examples of which are $A_{+++}A_{---}$, $A_{+++}A_{+--}$, $A_{++-}^2A_{+--}A_{---}$, and $A_{+++}^3A_{---}$. This gives rise to 30 homogeneous polynomials which are invariant under $\mathbf{R}^{(1)}$, and either invariant or changing sign under $\mathbf{R}^{(2)}$.

It is shown above that fundamental invariants for the general $O(2)$ gauge transformation (2) can be constructed in a straightforward fashion. We now specialise the consideration to the symmetric vertex model (4) for which the situation is considerably simpler.

Using (4) and (5), we have

$$A_{\sigma_1 \dots \sigma_p} \equiv A_p(t) = \sum_{s=0}^p i^s W_p(s) \sum_{\alpha=0}^m \binom{m}{\alpha} \sum_{\beta=0}^n \binom{n}{\beta} (-1)^\beta \delta_{\mathbf{K}_r}(\alpha + \beta, s) \quad (9)$$

where $t \equiv \sigma_1 + \sigma_2 + \dots + \sigma_p = \pm p, \pm(p-2), \dots, \pm 1$ or 0 ; $m \equiv (p+t)/2$, $n \equiv (p-t)/2$, and $\delta_{\mathbf{K}_r}$ is the Kronecker delta function. Note that the coefficient of $W_p(s)$ in (9) is the coefficient of z^s in the expansion of $(1+iz)^m(1-iz)^n$. It follows that (8) becomes

$$\tilde{A}_{\sigma_1 \dots \sigma_p}^{(l)} \equiv \tilde{A}_p^{(l)}(t) = \begin{cases} e^{i\theta} A_p(t) & l = 1 \\ e^{i\theta} A_p^*(t) & l = 2. \end{cases} \quad (10)$$

It is straightforward to write down the explicit expressions of $A_p(t)$ using (9). Adopting the notation [4,8] of denoting vertex weights by a, b, c, \dots such that a is the weight of vertices having no incident bonds, b the weight of vertices having one incident bond, etc, we find for $p = 2, 3, \dots, 6$

$$\begin{aligned} A_2(\pm 2) &= a - c \pm 2ib & A_2(0) &= a + c \\ A_3(\pm 3) &= a - 3c \pm i(3b - d) & A_3(\pm 1) &= a + c \pm i(b + d) \\ A_4(\pm 4) &= a - 6c + e \pm 4i(b - d) & A_4(\pm 2) &= a - e \pm 2i(b + d) \\ A_4(0) &= a + 2c + e \\ A_5(\pm 5) &= a - 10c + 5e \pm i(5b - 10d + f) & & (11) \\ A_5(\pm 3) &= a - 2c - 3e \pm i(3b + 2d - f) \\ A_5(\pm 1) &= a + 2c + e \pm i(b + 2d + f) \\ A_6(\pm 6) &= a - 15c + 15e - g \pm 2i(3b - 10d + 3f) \\ A_6(\pm 4) &= a - 5c - 5e + g \pm 4i(b - f) \\ A_6(\pm 2) &= a + c - e - g \pm 2i(b + 2d + f) \\ A_6(0) &= a + 3c + 3e + g. \end{aligned}$$

As dictated by (10), the fundamental invariants for each p are now constructed by forming products of $A_p(t)$, such that the sum of all t variables in the product vanishes. Adopting the notation $(t) \equiv A_p(t)$ for each p , we find the following fundamental invariants for $p = 2, 3, \dots, 6$:

$$\begin{aligned}
 p = 2: & \quad (2)(-2); \quad (0) \\
 p = 3: & \quad (3)(-3); \quad (1)(-1); \quad (3)(-1)^3, (-3)(1)^3 \\
 p = 4: & \quad (4)(-4); \quad (2)(-2); \quad (0); \quad (4)(-2)^2, (-4)(2)^2 \\
 p = 5: & \quad (5)(-5); \quad (3)(-3); \quad (1)(-1); \quad (5)(-1)^5, (-5)(1)^5; \quad (3)(-1)^3, (-3)(1)^3; \\
 & \quad (5)(-3)(-1)^2, (-5)(3)(1)^2; \quad (5)(1)(-3)^2, (-5)(-1)(3)^2; \\
 & \quad (5)^2(-1)(-3)^3, (-5)^2(1)(3)^3; \quad (5)^3(-3)^5, (-5)^3(3)^5 \\
 p = 6: & \quad (6)(-6); \quad (4)(-4); \quad (2)(-2); \quad (0); \quad (6)(-2)(-4), (-6)(2)(4); \\
 & \quad (4)(-2)^2, (-4)(2)^2; \quad (6)(-2)^3, (-6)(2)^3; \quad (6)(2)(-4)^2, (-6)(-2)(4)^2; \\
 & \quad (6)^2(-4)^3, (-6)^2(4)^3.
 \end{aligned}$$

Since (t) and $(-t)$ are complex conjugates of each other, the fundamental invariants always occur in conjugate pairs (except (0) and $(t)(-t)$ which are real), and we can consider the real and imaginary parts individually. Both the real and imaginary parts are invariant under $\mathbf{R}^{(1)}$, and the real parts are invariants and the imaginary parts change sign under $\mathbf{R}^{(2)}$. For $p = 3$, e.g., there are four polynomials:

$$\begin{aligned}
 I_1 &= (3)(-3) = (a - 3c)^2 + (3b - d)^2 \\
 I_2 &= (1)(-1) = (a + c)^2 + (b + d)^2 \\
 I_3 &= \text{Re}(3)(-1)^3 = \text{Re}\{[a - 3c + i(3b - d)][a + c - i(b + d)]^3\} \\
 I_4 &= \text{Im}(3)(-1)^3 = \text{Im}\{[a - 3c + i(3b - d)][a + c - i(b + d)]^3\}
 \end{aligned} \tag{12}$$

for which $I_1, I_2,$ and I_3 are invariant under both $\mathbf{R}^{(1)}$ and $\mathbf{R}^{(2)}$, and I_4 is invariant under $\mathbf{R}^{(1)}$ while changing sign under $\mathbf{R}^{(2)}$.

We have considered the $O(2)$ gauge transformation for a general two-state vertex model on an arbitrary lattice, and constructed its fundamental algebraic invariants. For the symmetric vertex model on a lattice of coordination number p , our analysis shows that there are, respectively, 2, 4, 5, 15, and 14 fundamental algebraic invariants for $p = 2, 3, 4, 5,$ and 6 . These invariants are explicitly given in (11). In the case of $p = 3$, we have verified that the fundamental invariants P, Q, P_1, P_2 and P_3 obtained previously [8,9] can indeed be expressed in terms of those in (12). The relations are $P = (9I_2 - I_1)/8, Q = (I_1 - I_2)/8, P_1 = I_4/4, P_2 = (72I_3 - I_1^2 + 30I_1I_2 + 27I_2^2)/64,$ and $P_3 = (-8I_3 - I_1^2 + 6I_1I_2 + 3I_2^2)/64$. Note that P_1 changes sign under $\mathbf{R}^{(2)}$, a fact previously observed [8]. For $p = 4$, we have also verified that the five fundamental invariants deduced from the ones obtained by Hijmans *et al* [6,7] can be expressed in terms of those in (11). The same set of $p = 4$ fundamental invariants have also been obtained, after considerable algebraic manipulation, by extending the analyses of Wu *et al* [8] and Gwa [9], but the method of Gwa no longer retains its simplicity for $p = 3$. Both methods, however, become almost intractable for $p > 4$.

Finally, we point out the existence of *syzygies*, polynomial relations between the linearly independent invariants. We have seen that all invariants for a given p are

products of $p+1$ polynomials $A_p(t)$. It follows that there must exist relations, or syzygies, among these invariants, if the number of invariants exceeds p . Explicit expressions of syzygies are usually very difficult to construct, but they are easily identified in the present formulation. For $p=3$ and 4, e.g., the numbers of fundamental invariants are, respectively, 4 and 5, and hence there is one syzygy in each case. Explicitly, we find

$$\begin{aligned} [(3)(-1)^3][(-3)(1)^3] &= [(3)(-3)][(1)(-1)]^3 && \text{for } p=3 \\ [(4)(-2)^2][(-4)(2)^2] &= [(4)(-4)][(2)(-2)]^2 && \text{for } p=4. \end{aligned} \quad (13)$$

Similarly, there are ten syzygies for $p=5$ and eight for $p=6$; all can be similarly constructed.

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