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LETTER TO THE EDITOR

The vicious neighbour problem

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Abstract. We compute the probability that a person will survive a shootout. The shootout involves N persons randomly placed in a d-dimensional space, each firing a single shot and killing his nearest neighbour with a probability p. We present a formulation which gives $P_N(p)$, the probability that a given person will survive, as a polynomial of p containing a finite number of terms. The coefficients appearing in the polynomial are explicitly evaluated for $d=1$ and $d=2$ in the limit of $N\to\infty$ to yield exact expressions for $P_\infty(p)$. In particular, $P_\infty(1)$ gives the probability that a given particle is *not* the nearest neighbour of any other particle in a classical ideal gas, and we further determine $P_\infty(1)$ for $d=3, 4$ and 5 using Monte Carlo simulations.

Consider N persons placed randomly in a bounded d-dimensional space. At a given instance, each person shoots, and kills, his nearest neighbour (called vicious neighbours) with a probability p. What is the fraction of persons who will survive the shootout in the limit of $N\to\infty$ and neglecting boundary corrections?

This problem of vicious neighbours, first posed by Abilock (1967) for $p=1$, has remained unsolved for almost two decades. The $d=2$ version of the $p=1$ problem re-appeared recently as a puzzle for which a prize was posted (Morris 1986, 1987). In this letter we present a solution to the general p problem for any spatial dimension d. More precisely, we present a formulation which gives $P_N(p)$, the fraction of persons who will survive the shootout, as a finite polynomial in p. We further show that coefficients of the polynomial are given in terms of finite-dimensional integrals in the limit of $N\to\infty$. For $d=1, 2$ these integrals are relatively simple and are explicitly evaluated to yield exact expressions for $P_\infty(p)$. For three and higher dimensions we compute $P_\infty(1)$ using independent Monte Carlo simulations.

We first summarise our findings for $p=1$, the problem originally proposed by Abilock (1967),

$$\begin{aligned} P_\infty(1) &= \tfrac{1}{4} && \text{for } d=1 \\ &= 0.284\,051\ldots && \text{for } d=2 \\ &= 0.303\ldots && \text{for } d=3 \text{ (Monte Carlo result)} \\ &= 0.318\ldots && \text{for } d=4 \text{ (Monte Carlo result)} \\ &= 0.328\ldots && \text{for } d=5 \text{ (Monte Carlo result).} \end{aligned} \tag{1}$$

Explicit expressions for $P_\infty(p)$ for $d=1$ and $d=2$ are given by (14) and (39).

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It is convenient to regard the N persons as being particles in a many-body system. Then $P_N(p)$ is the probability that a given particle will survive the shootout, averaged over all particle configurations. As an example of possible application, $P_\infty(1)$ now gives the probability that a given particle is *not* the nearest neighbour of any other particle in a classical ideal gas. Our goal is to compute the thermodynamic limit

$$P_\infty(p) = \lim_{N\to\infty} P_N(p). \tag{2}$$

Number the particles from 0 to $N-1$ and consider the survival of particle 0. Each particle (other than 0) can be in one of two 'states': that it either kills, or does not kill, particle 0. Regard the occurrence of these two states as a probabilistic event and denote the probability that n particles, numbered $j_1, j_2, \dots j_n$, *all* shoot (and kill) particle 0, regardless of the states of the other $N-n-1$ particles, by

$$p(j_1, j_2, \dots, j_n) = p^n w(j_1, j_2, \dots, j_n) \qquad n = 1, 2, \dots, N-1 \tag{3}$$

where $w(j_i, j_2, \dots, j_n)$ is the probability that the n particles $j_1, j_2, \dots, j_n$ will find 0 as their common nearest neighbour. Then as a consequence of an identity in probability theory (Whitney 1932) we can express $P_N(p)$, the probability that all $N-1$ particles are in one state (of not killing 0), as a linear combination of $p(j_1, j_2, \dots j_n)$, the probability that the n particles $j_1, j_2, \dots, j_n$ are in the other state (*all* killing 0), as follows:

$$\begin{aligned} P_N(p) = 1 - \sum_{j=1}^{N-1} p(j) + \sum_{1\le j_1<j_2\le N-1} p(j_1, j_2) + \dots \\ + \sum_{1\le j_1<\dots<j_n\le N-1} (-1)^n p(j_1, \dots, j_n) + \dots \\ + (-1)^{N-1} p(1, 2, \dots, N-1). \end{aligned} \tag{4}$$

Since all particles $1, 2, \dots, N-1$ are equivalent in the consideration of the survival of particle 0, we can write (4) as

$$P_N(p) = 1 + \sum_{n=1}^{N-1} C_n(-p)^n \tag{5}$$

where

$$C_n = \binom{N-1}{n} w(1, 2, \dots, n). \tag{6}$$

The intriguing fact which allows the problem to be exactly soluble is that

$$w(1, 2, \dots, n) = 0 \qquad \text{for } n > n_d \tag{7}$$

where n_d is a finite integer whose value depends on the spatial dimension d. That is, no more than n_d particles can simultaneously find particle 0 as their common nearest neighbour. It is easy to see this for $d = 1$ since, for particles arranged on a line, there can be at most two particles having particle 0 as their nearest neighbours. Thus, we have $n_1 = 2$. In the case of $d = 2$ we assume there exist n particles, numbered $1, 2, \dots, n$, all having particle 0 as their nearest neighbours (cf figure 1). For particles 1 and 2 to have particle 0 as their nearest neighbour, we must have $r_{12} > r_{01}$ and r_{02}, where r_{ij} is the distance between particles i and j, and, consequently, $\theta_1 > \pi/3$, where θ_1 is the angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Similarly we find $\theta_i > \pi/3$, $i = 2, 3, \dots, m$, for the other $n-1$ angles. The sum rule $\sum_{j=1}^{n} \theta_j = 2\pi$ now implies that $n \le 5$ and hence $n_2 = 5$.

Generally, the integer n_d for $d \ge 2$ is bounded by the maximum number of d-dimensional regular $(d+1)$-polyhedra that can be fitted together such that they all

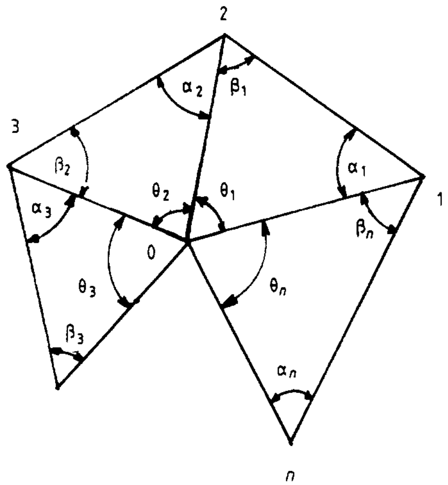

Figure 1. Configuration showing that n particles have particle 0 as their common nearest neighbour.

have the origin as a common vertex and there is still room for the polyhedra to rotate slightly about the origin without spoiling the fit. In three dimensions one can fit at most 22 regular tetrahedra at the origin without exhausting the whole solid angle 4π (Coxeter 1969). It follows that n_3 cannot be greater than 22.

Up to this point we have regarded N finite and have not considered the fact that the region confining the N particles is bounded. Let Ω be the volume of the region. We shall take the thermodynamic limit $N \to \infty$, $\Omega \to \infty$ with the density $\rho = N/\Omega$ held constant, a limit we denote by $N \to \infty$ for brevity. While there is no intrinsic length in the problem, so that the final result is expected to be independent of ρ, the introduction of the density ρ is a convenient tool which enables us to take the limit appropriately.

It is relatively easy to see that

$$\lim_{N\to\infty} C_1 \equiv (N-1)w(1) = 1 \qquad \text{for all } d. \tag{8}$$

This is so since $w(1)$, the probability that r_{01} is the shortest among the $N-1$ distances r_{i1}, $i = 0, 2, 3, \ldots, N-1$, is $1/(N-1)$ after the boundary corrections are ignored.

Consider next the evaluation of $C_2 = \binom{N-1}{2} w(1, 2)$, where $w(1, 2)$ is the probability that both particles 1 and 2 have particle 0 as their nearest neighbours. For this to happen we must have $r_1, r_2 < r_{12}$ and, in addition, $r_1 < r_{i1}$, $r_2 < r_{i2}$, for $i = 3, 4, \ldots, N-1$. Let $S_2(r_1, r_2, \Omega)$ be the volume common to Ω and the union of two spheres centred at $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ with respective radii r_2 and r_3 (thus both passing through the origin). Then, since $N-3$ particles must stay outside S_2, we have

$$C_2 = \frac{(N-1)(N-2)}{2!} \int_{r_1, r_2 < r_{12}} \frac{\mathrm{d}\boldsymbol{r}_1}{\Omega} \frac{\mathrm{d}\boldsymbol{r}_2}{\Omega} \left(1 - \frac{S_2(\boldsymbol{r}_1, \boldsymbol{r}_2, \Omega)}{\Omega}\right)^{N-3}. \tag{9}$$

Taking the thermodynamic limit now leads to

$$\begin{aligned} c_2 \equiv \lim_{N\to\infty} C_2 &= \frac{1}{2!} \rho^2 \int_{r_1, r_2 < r_{12}} \mathrm{d}\boldsymbol{r}_1 \, \mathrm{d}\boldsymbol{r}_2 \exp[-\rho \bar{V}_2(\boldsymbol{r}_1, \boldsymbol{r}_2)] \\ &= \frac{1}{2!} \int_{r_1, r_2 < r_{12}} \mathrm{d}\boldsymbol{r}_1 \, \mathrm{d}\boldsymbol{r}_2 \exp[-V_2(\boldsymbol{r}_1, \boldsymbol{r}_2)] \end{aligned} \tag{10}$$

where $V_2(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is the volume occupied by the two aforementioned spheres, a situation shown in figure 2 for $d=2$.

Proceeding in the same fashion we obtain, quite generally,

$$c_n \equiv \lim_{N\to\infty} C_n = \frac{1}{n!}\int_{r_i<r_{ij}} \mathrm{d}\boldsymbol{r}_1 \ldots \mathrm{d}\boldsymbol{r}_n \exp[-V_n(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n)] \tag{11}$$

where $V_n(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n)$ is the volume occupied by n spheres centred at $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n$ and intersecting at the origin. Here, the restrictions $r_i < r_{ij}$, $i \neq j = 1, 2, \ldots, n$ ensure that particle 0 is the common nearest neighbour of particles $i = 1, 2, \ldots, n$. Finally, the survival probability $P_\infty(p)$ is obtained by combining (2) and (5) as

$$P_\infty(p) = 1 - p + c_2p^2 - c_3p^3 + \ldots + c_{n_d}(-p)^{n_d} \tag{12}$$

where c_n is given by (11).

In one dimension we have $n_1 = 2$ and hence, from (10) and (12),

$$P_\infty(p) = 1 - p + \tfrac{1}{2}p^2 \int_{x_1,x_2<x_{12}} \mathrm{d}x_1\, \mathrm{d}x_2 \exp[-V_2(x_1, x_2)] \tag{13}$$

where $x_{12} = |x_1 - x_2|$ and the two integrations range from $-\infty$ to ∞. Now the restriction $x_1, x_2 < x_{12}$ implies that x_1 and x_2 must have opposite signs and, therefore, $V_2(x_1, x_2) = 2(|x_1| + |x_2|)$. The integrations in (13) are thus simply carried out, yielding

$$P_\infty(p) = 1 - p + 2\times\tfrac{1}{2}p^2 \int_{-\infty}^{0} \exp(-2|x_1|)\, \mathrm{d}x_1 \int_0^\infty \exp(-2|x_2|)\, \mathrm{d}x_2 = 1 - p + \tfrac{1}{4}p^2 \tag{14}$$

which, for $p = 1$, reduces to $\frac{1}{4}$, the result quoted in (1).

We now evaluate (12) in two dimensions, where $n_2 = 5$, term by term by explicitly reducing the coefficients c_i into quadratures.

The integrand in (11) is invariant under permutations of the n vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$. This permits us to focus on a particular arrangement of the vectors, such as the one shown in figure 1, which occurs $(n-1)!$ times. The key to the reduction of (11) lies in the observation that, for r_i fixed, the constraints r_i, $r_{i+1} < r_{i,i+1}$ effectively restrict $t_{i+1} \equiv r_{i+1}/r_i$ to range from $t_{\min}(\theta_i)$ to $t_{\max}(\theta_i)$, where θ_i is the angle between the vectors $\boldsymbol{r}_i$ and $\boldsymbol{r}_{i+1}$, and with

$$\{t_{\min}(\theta_i), t_{\max}(\theta_i)\} = \begin{cases} \{2\cos\theta_i, (2\cos\theta_i)^{-1}\} & \text{if } \pi/3 < \theta_i < \pi/2 \\ \{0, \infty\} & \text{if } \pi/2 < \theta_i < 3\pi/2. \end{cases} \tag{15}$$

It is therefore important to treat the cases of $\theta_i < \frac{1}{2}\pi$ and $\theta_i > \frac{1}{2}\pi$ separately.

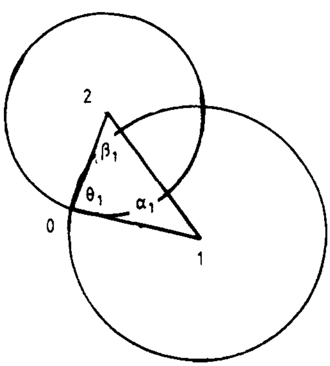

Figure 2. Configuration showing the area occupied by two circles.

Consider first the integrations in (11) over the directions of $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$. We decompose the phase space of these angular integrations into regions according to whether each of the n angles θ_i is greater or less than $\frac{1}{2}\pi$. This decomposition is facilitated by assigning a two-valued variable σ_i to the angle θ_i such that $\sigma_i = 0$ if $\theta_i > \frac{1}{2}\pi$ and $\sigma_i = 1$ if $\theta_i < \frac{1}{2}\pi$. Let $I(\sigma_1, \sigma_2, \ldots, \sigma_n)$ be the contribution to c_n with $\{\theta_1, \ldots, \theta_n\}$ in the range specified by $\sigma_1, \sigma_2, \ldots, \sigma_n$. Clearly, I is invariant under cyclic permutations of its arguments, i.e. $I(\sigma_1, \ldots, \sigma_n) = I(\sigma_2, \ldots, \sigma_n, \sigma_1)$, etc. After taking this degeneracy into consideration, we find

$$\begin{aligned} c_2 &= I(0,0) + 2I(1,0) \\ c_3 &= I(0,0,0) + 3I(1,0,0) + 3I(1,1,0) \\ c_4 &= 4I(1,0,0,0) + 4I(1,1,0,0) + 2I(1,0,1,0) + 4I(1,1,1,0) \\ c_5 &= 5I(1,1,1,1,0) + I(1,1,1,1,1) \end{aligned} \tag{16}$$

where we have used the fact that each θ_i must range between $\frac{1}{3}\pi$ and $\frac{5}{3}\pi$ and thus, e.g., there can be only two ways to fit five angles in the case of $n = 5$.

Consider next the n radial integrations over $\mathrm{d}r_1, \ldots, \mathrm{d}r_n$. Using the fact that the volume $V_n(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n)$ is homogeneous and quadratic in $r_1, \ldots, r_n$, we can write for each term in (16)

$$V_n = r_1^2 V_n(t_2, t_3, \ldots, t_n; \theta_1, \theta_2, \ldots, \theta_{n-1}) \tag{17}$$

where $t_i \equiv r_i/r_{i-1}$. Thus, after introducing the variables t_i, $i = 2, \ldots, n$, into the integrand, the integration over r_1 can be carried out, yielding a factor

$$2\pi \int_0^\infty r_1^{2n-1}\,\mathrm{d}r_1 \exp[-r_1^2 V_n(t_2, t_3, \ldots, t_n; \theta_1, \theta_2, \ldots, \theta_{n-1})] = \pi(n-1)!(V_n)^{-n}. \tag{18}$$

Thus we find, for c_2,

$$I(0,0) = \pi \int_{\pi/2}^{3\pi/2} \mathrm{d}\theta_1 \int_0^\infty t_2\,\mathrm{d}t_2 [V_2(t_2; \theta_1)]^{-2} = 0.258\,572\,168\ldots \tag{19}$$

$$I(1,0) = \pi \int_{\pi/3}^{\pi/2} \mathrm{d}\theta_1 \int_{2\cos\theta_1}^{(2\cos\theta_1)^{-1}} t_2\,\mathrm{d}t_2 [V_2(t_2; \theta_1)]^{-2} = 0.028\,880\,652\ldots \tag{20}$$

where

$$V_2(t_2; \theta_1) = z_1 + t_2^2 z_2 \tag{21}$$

and

$$\begin{aligned} z_1 &= \pi - \alpha_1 + \tfrac{1}{2}\sin(2\alpha_1) \\ z_2 &= \pi - \beta_2 + \tfrac{1}{2}\sin(2\beta_2) \end{aligned} \tag{22}$$

α_1 and β_2 being the angles shown in figure 2 and given by, with $i = 1$,

$$\begin{aligned} \sin\alpha_i &= t_{i+1} \sin\theta_i (1 + t_{i+1}^2 - 2t_{i+1}\cos\theta_i)^{-1/2} \\ \sin\beta_{i+1} &= \sin\theta_i (1 + t_{i+1}^2 - 2t_{i+1}\cos\theta_i)^{-1/2}. \end{aligned} \tag{23}$$

Substitution of (19) and (20) into (16) now gives

$$c_2 = 0.316\,3335\ldots. \tag{24}$$

In a similar fashion we find, for c_3,

$$I(0,0,0)=\frac{2\pi}{3}\int_{\pi/2}^{\pi}\mathrm{d}\theta_1\int_{\pi/2}^{3\pi/2-\theta_1}\mathrm{d}\theta_2\int_0^\infty t_2^3\,\mathrm{d}t_2\int_0^\infty t_3\,\mathrm{d}t_3[V_3]^{-3}=0.011\,207\,724\ldots \tag{25}$$

$$I(1,0,0)=\frac{2\pi}{3}\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_1\int_{\pi/2}^{3\pi/2-\theta_1}\mathrm{d}\theta_2\int_{2\cos\theta_1}^{(2\cos\theta_1)^{-1}} t_2^3\,\mathrm{d}t_2\int_0^\infty t_3\,\mathrm{d}t_3[V_3]^{-3}$$
$$=0.005\,621\,972\ldots \tag{26}$$

$$I(1,1,0)=\frac{2\pi}{3}\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_1\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_2\int_{2\cos\theta_1}^{(2\cos\theta_1)^{-1}} t_2^3\,\mathrm{d}t_2\int_{2\cos\theta_2}^{(2\cos\theta_2)^{-1}} t_3\,\mathrm{d}t_3[V_3]^{-3}$$
$$=0.001\,168\,842\ldots \tag{27}$$

where, quite generally,

$$V_n=z_1+t_2^2z_2+\ldots+(t_2\ldots t_n)^2z_n \qquad (n=3,4,5) \tag{28}$$

$$z_i=\pi-\alpha_i+\tfrac{1}{2}\sin 2\alpha_i-\beta_{i-1}+\tfrac{1}{2}\sin 2\beta_{i-1} \qquad (i=1,2,\ldots,n). \tag{29}$$

Here the angles α_i and β_i, shown in figure 1, are related to the integration variables through (23), with $\theta_n=2\pi-\theta_1-\theta_2-\ldots-\theta_{n-1}$, and subject to the constraints $\beta_0\equiv\beta_n$. Special care must be taken for $n=3$, a situation shown in figure 3, for which we must set $\alpha_2=\beta_2=0$ if $\theta_2>\pi$ and $\alpha_3=\beta_3=0$ if $\theta_1+\theta_2<\pi$.

Substitution of (25)-(27) into (16) now gives

$$c_3=0.032\,9390\ldots. \tag{30}$$

Similarly, for c_4 we find

$$I(1,0,0,0)=\frac{3\pi}{2}\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_1\int_{\pi/2}^{\pi-\theta_1}\mathrm{d}\theta_2\int_{\pi/2}^{3\pi/2-\theta_1-\theta_2}\mathrm{d}\theta_3$$
$$\times\int_{2\cos\theta_1}^{(2\cos\theta_1)^{-1}} t_2^5\,\mathrm{d}t_2\int_0^\infty t_3^3\,\mathrm{d}t_3\int_0^\infty t_4\,\mathrm{d}t_4[V_4]^{-4} \tag{31}$$

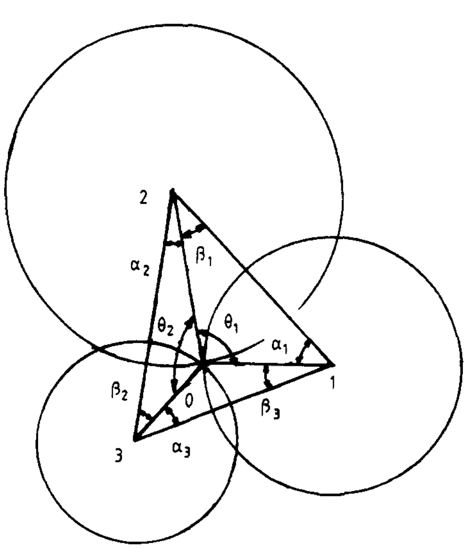

Figure 3. Configuration showing the area occupied by three circles intersecting at one point with $\theta_1+\theta_2>\pi$ and $\theta_2<\pi$.

$$I(1,1,0,0)=\frac{3\pi}{2}\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_1\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_2\int_{\pi/2}^{3\pi/2-\theta_1-\theta_2}\mathrm{d}\theta_3\int_{2\cos\theta_1}^{(2\cos\theta_1)^{-1}}t_2^5\,\mathrm{d}t_2$$
$$\times\int_{2\cos\theta_2}^{(2\cos\theta_2)^{-1}}t_3^3\,\mathrm{d}t_3\int_0^\infty t_4\,\mathrm{d}t_4[V_4]^{-4} \qquad (32)$$

$$I(1,0,1,0)=\frac{3\pi}{2}\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_1\int_{\pi/2}^{7\pi/6-\theta_1}\mathrm{d}\theta_2\int_{\pi/3}^{\min\{3\pi/2-\theta_1-\theta_2,\pi/2\}}\mathrm{d}\theta_3$$
$$\times\int_{2\cos\theta_1}^{(2\cos\theta_1)^{-1}}t_2^5\,\mathrm{d}t_2\int_0^\infty t_3^3\,\mathrm{d}t_3\int_{2\cos\theta_3}^{(2\cos\theta_3)^{-1}}t_4\,\mathrm{d}t_4[V_4]^{-4} \qquad (33)$$

$$I(1,1,1,0)=\frac{3\pi}{2}\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_1\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_2\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_3$$
$$\times\int_{2\cos\theta_1}^{(2\cos\theta_1)^{-1}}t_2^5\,\mathrm{d}t_2\int_{2\cos\theta_2}^{(2\cos\theta_2)^{-1}}t_3^3\,\mathrm{d}t_3\int_{2\cos\theta_3}^{(2\cos\theta_3)^{-1}}\mathrm{d}t_4[V_4]^{-4}. \qquad (34)$$

Numerical evaluation of (31)-(34) yields

$$c_4=0.000\,6575\,\ldots. \qquad (35)$$

In the evaluation of c_5 we have

$$I(1,1,1,1,0)=\frac{24\pi}{5}\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_1\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_2\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_3\int_{\min\{\pi/3,3\pi/2-\theta_1-\theta_2-\theta_3\}}^{3\pi/2-\theta_1-\theta_2-\theta_3}\mathrm{d}\theta_4$$
$$\times\int_{2\cos\theta_1}^{(2\cos\theta_1)^{-1}}t_2^7\,\mathrm{d}t_2\int_{2\cos\theta_2}^{(2\cos\theta_2)^{-1}}t_3^5\,\mathrm{d}t_3\int_{2\cos\theta_3}^{(2\cos\theta_3)^{-1}}t_4^3\,\mathrm{d}t_4$$
$$\times\int_{2\cos\theta_4}^{(2\cos\theta_4)^{-1}}t_5\,\mathrm{d}t_5[V_5]^{-5} \qquad (36)$$

$$I(1,1,1,1,1)=\frac{24\pi}{5}\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_1\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_2\int_{\pi/3}^{\pi/2}\mathrm{d}\theta_3$$
$$\times\int_{\min\{\max\{\pi/3,3\pi/2-\theta_1-\theta_2-\theta_3\},5\pi/3-\theta_1-\theta_2-\theta_3\}}^{\min\{\pi/2,5\pi/3-\theta_1-\theta_2-\theta_3\}}\mathrm{d}\theta_4\int_{2\cos\theta_1}^{(2\cos\theta_1)^{-1}}t_2^7\,\mathrm{d}t_2$$
$$\times\int_{2\cos\theta_2}^{(2\cos\theta_2)^{-1}}t_3^5\,\mathrm{d}t_3\int_{2\cos\theta_3}^{(2\cos\theta_3)^{-1}}t_4^3\,\mathrm{d}t_4\int_{\max\{2\cos\theta_4,2\cos\theta_5\}}^{\min\{(2\cos\theta_4)^{-1},(2\cos\theta_5)^{-1}\}}t_5\,\mathrm{d}t_5[V_5]^{-5}. \qquad (37)$$

Numerical evaluation of (36) and (37) gives

$$c_5=0.000\,0010\,\ldots. \qquad (38)$$

Finally, upon combining (12), (24), (30), (35) and (38), we obtain

$$P_\infty(p)=1-p+0.316\,3335p^2-0.032\,9390p^3+0.000\,6575p^4-0.000\,0010p^5 \qquad (39)$$

which, for $p=1$, reduces to 0.284 051 ..., the result quoted in (1).

The evaluation of $P_\infty(p)$ given by (12) can, in principle, be carried for any d. For $d=3$, for example, we replace circles by spheres in the above consideration and it is necessary to evaluate 21 terms at most in (12), each of which is a multidimensional integral. However, these integrals are fairly complicated and, instead, we have used independent Monte Carlo simulations to obtain estimates of $P_\infty(1)$. Simulations on a VAX computer for systems consisting of up to 10 000 particles yield the results in (1). To check the accuracy of our simulations, we applied the same procedure to the $d = 2$ system and obtained the number $P_\infty(1) = 0.284 \pm 0.003$, in excellent agreement with the exact result (39).

Note added. After the submission of this letter, Veit Elser and Friend Kierstead Jr have called our attention to the known fact that $n_3 = 12$. Dr Elser also provided upper bounds on n_d for d up to 24.

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