An infinite-range bond percolation[a)]

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A bond percolation on a lattice in which all pairs of vertices are connected is considered. The percolation problem is treated by carrying out the Kasteleyn-Fourtuin formulation of taking the $q = 1$ limit of a related q-component Potts model; the latter is exactly solved. For a lattice of N sites, an average of pN occupied bonds and N large, it is found that the system is percolating for $p > 1/2$. Closed-form expression is also obtained for the moment-generating function of the cluster size. The analysis yields the mean-field exponents $\beta = \gamma = \gamma' = 1$.

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It is customary to regard the Bethe-lattice solution of the percolation problem as its mean-field approximation [1]. While the Bethe-lattice consideration has led to yield the mean-field exponents, it has not been very helpful in providing an useful picture in understanding the mean-field nature of the percolation transition. In this paper we consider a model of a bond percolation, which is an extension of the usual mean-field spin models. By solving this percolation problem exactly we have a picture of a mean-field percolation which is on the same footing as those of the spin systems.

It is well-known that a meaningful mean-field model for spin systems is one in which all spins interact with equal strength of the order of 1/N, N being the total number of spins [2]. In the same spirit we therefore picture a mean-field model for bond percolation as a percolation process is which all pairs of sites are connected by bonds with an independent probability, also of the order of 1/N. Specifically, consider a system of N sites subject to a percolation process in which every pair of sites can be connected by a bond with a probability 2p/N; then on the average there are pN bonds. Thus this is a percolation with long-range interations. It is known that such a percolation describes the problem of random graphs [3], and can be treated using a purely probabilistic approach [4]. Here we utilize the method of statistical physics by first considering a related Potts model [5]. The Potts model is then solved to give results on the percolation problem.

Consider a system of N q-state Potts spins whose Hamiltonian reads

$$H = \frac{K}{N}\sum_{<i,j>}\delta(\sigma_i,\sigma_j) + L\sum_i \delta(\sigma_i,1) \qquad (1)$$

where we have taken $kT = -1$, $\sigma_i = 1,2,\ldots,q$ denotes the spin states at the ith site, $i = 1,2,\ldots,N$ and δ is the Kronecker delta function. The summations $<i,j>$ are taken between all pairs of spins.

It is well-known [5,6] that the $q = 1$ limit of a Potts model generates a percolation. Particularly the Potts Hamiltonian (1) generates the long-range percolation under consideration with the following cluster-size generating function [6]:

$$G(L) = \frac{1}{N}\left\langle \sum_c \exp(-Ls_c)\right\rangle$$

$$= \left[\frac{\partial}{\partial q} A(q;K,L)\right]_{q=1} \qquad (2)$$

where

$$2p/N = 1 - e^{-K/N} \qquad (3)$$

Here s_c is the number of sites in a cluster c, $A(q;K,L)$ is the free energy of (1).

To compute $A(q;K,L)$, let $x_i, i=1,2,\ldots,q$, denote the fraction of spins in the ith spin state. Then, in the limit of $N\to\infty$, we have

$$A(q;K,L) = \max_{\{x_i\}}[E(x_i) - \sum_i x_i \ln x_i] \qquad (4)$$

where

$$E(x_i) = \frac{K}{2}\sum x_i^2 + Lx_1 \qquad (5)$$

is the "energy" of the system computed from (1) and subject to $\sum x_i = 1$.

It is now straightforward to compute $A(q;K,L)$ from (4) and (5). The result yields [7]

$$G(L) = 1 - s_o - p(1-s_o)^2 \qquad (6)$$

where s_o is determined from

$$2ps_o + L + \ln(1-s_o) = 0 \qquad (7)$$

Quantities of interest in percolation can now be computed by taking the respective derivatives of $G(L)$. Thus, we find the percolation probability, $P(p)$, and the mean cluster size, $S(p)$, to be given by [6]

$$P(p) = 1 + G'(0+) = s_o \qquad (8)$$

$$S(p) = G''(0+) = (1-s_o)/(1-2p+2ps_o) \qquad (9)$$

Now, for $p<1/2$, (7) with $L=0$ has only one solution $s_o=0$, hence $P(p)=0$ identically. For $p>1/2$, however, a second solution $s_o>0$ arises and there is a nonzero percolation probability $P(p)=s_o$. The average cluster size is given by (9). Near the threshold we find

$$P(p) \simeq p - 1/2 \qquad (10)$$

$$S(p) \simeq |p - 1/2|^{-1} \qquad (11)$$

This leads to the mean-field sxponents $\beta=\gamma=\gamma'=1$, in agreement with the finding of [1]. We also note that the critical value of $p = 1/2$ concides with the finding of the purely probabilistic approach [4].

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