J. Phys. A: Math. Gen. 13 (1980) L303–L305. Printed in Great Britain

LETTER TO THE EDITOR

On the equivalence of the Ising model with a vertex problem†

F Y Wu

Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA and Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061, USA

Received 9 June 1980

Abstract. The planar Ising model in an external magnetic field and with multi-spin interactions is shown to be equivalent to a vertex problem. The equivalence can be extended to higher spins and higher dimensions.

In a recent paper Malakis (1979) reported the proof of an equivalence between a certain staggered vertex model and a zero-field Ising model. The equivalence has since been extended (Malakis 1980) to a wider class of vertex problems on the square lattice.

The purpose of this Letter is to point out that the result of Malakis (1979) is not new and, in fact, can be easily derived in a very general form to include the Ising model with *non-zero* magnetic field as well as other lattices. It can also be extended to higher spins and higher dimensions. In view of the rather elaborate effort required in Malakis' proof, it appears desirable to report on this development.

It was first noted by Wu and Lin (1975) that the square lattice Ising model in a non-zero magnetic field is reducible to a staggered vertex model. This equivalence, of which the result of Malakis (1979) is a special case, has also been reported in Wu (1978). The derivation given by Wu and Lin (1975) is very simple. We now explore its extensions.

Consider an *even* planar graph (or lattice) G which has an even number of edges incident at each vertex. Examples are the square, Kagomé and triangular lattices. Since the faces of G are bipartite, we may shade one set of the faces as shown in figure 1. Consider next an Ising model whose spins are located at the shaded faces of G and whose interactions, including external magnetic field and possibly multi-spin interactions, are decomposable into sums over the vertices of G. Specifically, let $\sigma_1 \ldots, \sigma_k$ be the spins surrounding a site of G. Then the Ising Hamiltonian $\mathscr{H}$ takes the form

$$\mathscr{H} = \sum_{\text{sites of } G} f(\sigma_1, \ldots, \sigma_k). \tag{1}$$

Under this circumstance, we have the following equivalence:

the Ising model (1) *is equivalent to a vertex model on* G.

The proof of this equivalence, which also serves to define the vertex model, is extremely simple. For a given spin configuration in the Ising model, encircle the (shaded) faces

† Supported in part by NSF Grant No DMR 78-18808.

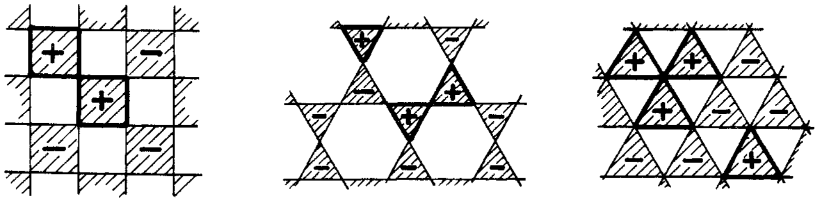

Figure 1. Examples of even lattices G and the associated Ising spins. (a) Quartic lattices. (b) Vertex model on the Kagomé lattice and the Ising model on the honeycomb lattice. (c) Vertex model on the triangular lattice and an Ising model with 1-, 2- and 3-spin interactions.

occupied by the '+' spins by drawing bonds along the edges (cf figure 1). This leads to a bond graph on G. It is clear that 2^k bond configurations can occur at a vertex of G surrounded by k spins. Conversely, using these 'allowed' bond configurations at each vertex, we can construct all possible bond graphs on G derived from the spin configurations. The mapping between the bond graphs and spin configurations is clearly one-to-one. Thus, the Ising partition function is identically that of a vertex model on G, provided that we take the vertex weights

$$\begin{aligned} \omega &= \exp[-f(\sigma_1, \ldots, \sigma_k)/kT] && \text{for allowed configurations} \\ &= 0 && \text{otherwise.} \end{aligned} \tag{2}$$

This completes the proof.

We now examine applications of this equivalence. For the quartic Ising model with nearest-neighbour interactions $-J = -kTK$ and magnetic field $-H = -kTL$, the lattice G is also quartic (figure 1(a)) and the vertex weights are

$$\begin{aligned} \omega &= \exp[K\sigma_1\sigma_2 + L(\sigma_1+\sigma_2)/z] && \text{for allowed configurations} \\ &= 0 && \text{otherwise.} \end{aligned} \tag{3}$$

Here $2^2 = 4$ configurations are allowed at each vertex and $z = 4$ is the coordination number of the lattice. This is the result reported by Wu and Lin (1975). Note that the resulting vertex weight (3) has a staggered feature since the geometry of the vertices rotates 90° from sublattice to sublattice.

Similarly, as shown in figure 1(b), a staggered vertex model on the Kagomé lattice is equivalent to the nearest-neighbour Ising model on the honeycomb lattice. The vertex weight is again given by (3) where we take $z = 3$. Now, if the Ising spins are chosen to be located at the unshaded faces of the Kagomé lattice, we arrive at an equivalence between the nearest-neighbour triangular Ising model and the staggered Kagomé vertex model (3) with $z = 6$. The extension of the above considerations to anisotropic interactions is obvious.

Of further interest is the fact that the vertex representation of an Ising model is not necessarily unique. A case of interest is the alternate representation of the triangular Ising net shown in figure 1(c). As seen from the figure, the lattice G is also triangular and has a *uniform* vertex weight. We can even include in the Ising model three-spin interactions $-J_3 = -kTK_3$ (among every three spins surrounding a vertex). For such an

Ising model, the equivalence now reads

$$\begin{aligned}\omega &= \exp[K(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)+K_3\sigma_1\sigma_2\sigma_3 \\ &\quad +L(\sigma_1+\sigma_2+\sigma_3)/3] \qquad \text{for allowed configurations} \\ &= 0 \qquad \text{otherwise.}\end{aligned} \tag{4}$$

The model (4) is a special case of the 32-vertex model considered by Sacco and Wu (1975). In fact, only 8 of the 32 bond configurations are allowed. These configurations are shown in figure 2 with the associated weights. Unfortunately, the weights do not satisfy the solubility conditions given by Sacco and Wu (1975), and hence the model is not soluble.

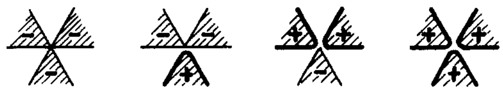

Figure 2. The eight allowed vertex configurations in figure 1(*c*) and the associated weights (4). Four of the configurations not shown are obtainable by 120° rotations.

Finally, we remark that our result is valid generally for any *even* graph *G*, which does not have to be regular. It should also be remarked that, instead of using bond graphs, the vertex model can also be described by arrow configurations. The mapping of bond graphs into arrow configurations can be carried out in a number of ways (Lieb and Wu 1972, Wu and Lin 1975). We shall not repeat these discussions as they are not essential to our result.

In conclusion, we have shown that the Ising model with an external field as well as multi-spin interactions can be formulated as a vertex problem. The consideration can be extended to higher spins and higher dimensions in an obvious way. For the spin-1 Ising model, for example, two kinds of bonds, say red and blue, are needed to encircle '+1' and '−1' spins. The resulting vertex model will then include all 'allowed' bond configurations of two colours. In three dimensions, the resulting vertex model will also be in three dimensions and defined by planes; the vertex weights are assigned according to the vertex plane configurations.

References

Lieb E H and Wu F Y 1972 *Phase transitions and Critical Phenomena* ed C Domb and M S Green (London: Academic) vol 1 p 331
Malakis A 1979 *J. Phys. A: Math. Gen.* **12** L275–9
—— 1980 *J. Phys. A: Math. Gen.* **13** L55–9
Sacco J E and Wu F Y 1975 *J. Phys. A: Math. Gen.* **8** 1780–7
Wu F Y 1978 *Ann. Israel Phys. Soc.* **2** 370–6
Wu F Y and Lin K Y 1975 *Phys. Rev.* B **12** 419–28