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LETTER TO THE EDITOR

Critical polarisation of the modified F model†

X Sun‡§ and F Y Wu||

‡ Department of Physics, Fudan University, Shanghai, People's Republic of China
and Department of Physics and Astronomy, Northwestern University, Evanson, Illinois 60201, USA¶

|| Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA

Abstract. It is shown that the modified F (MF) model in small electric field is related to a nearest-neighbour Ising model in an external magnetic field. This leads to the result that the small-field polarisation of the MF model is equal to the square of the magnetisation of the corresponding Ising model. Applying this equivalence to the critical isotherm we obtain the critical exponent relation $\delta_e = \frac{1}{2}(\delta_m - 1)$ and a similar expression relating the amplitude of the critical isotherm of the MF model to that of the nearest-neighbour Ising model. From this and the known value of $\delta_m = 15$, we obtain $\delta_e = 7$ confirming the scaling prediction.

The modified F (MF) model was introduced by Wu (1969) as an exactly soluble model of a phase transition. One interesting feature of the MF model is that, considered as a model of a ferroelectric, it is one of the few models for which many of the critical exponents can be exactly determined. In zero external electric field the MF model is equivalent to a *nearest-neighbour* Ising model (Wu 1969, Lieb and Wu 1972). This equivalence leads to the determination of many of the zero-field critical exponents (Brascamp *et al* 1973, Enting 1973, Enting and Gaunt 1974, Baxter and Kelland 1974). When the external electric field is non-zero, the Ising equivalence involves crossing interactions (Wu 1969, Lieb and Wu 1972) and the situation is more complicated. As a consequence, there has been no direct determination of the critical exponent δ_e of the MF model.

In this Letter we show that the difficulty of crossing interactions in the Ising equivalence can be circumvented when the external field is small. Specifically, we establish that the MF model in a small external electric field is equivalent to a nearest-neighbour Ising model in an external magnetic field. This resolves the difficulty of the crossing interactions, and permits us to relate the exponent δ_e to the corresponding magnetic exponent δ_m of the nearest-neighbour Ising model. Using the accepted value $\delta_m = 15$, we obtain $\delta_e = 7$, confirming the prediction of scaling (Enting and Gaunt 1974). We also obtain an expression relating the amplitude of the critical isotherm of the MF model to that of the nearest-neighbour Ising model.

Our analysis is based on a linked-graph expansion theorem established by one of us (Sun 1975). Since the theorem was originally developed for quantum spin systems, we restate the theorem here in a language suitable for application to Ising spins. We refer to Sun (1975) for details and the proof of the theorem.

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¶ Present address.

Theorem. Let κ be the Hamiltonian of an Ising system of N spins and

$$H_0 = \sum_i c_i \sigma_i \tag{1}$$

where c_i are constants. Let $f(\kappa)$ be the free energy per spin of the system, and

$$\langle \ldots \rangle_\kappa = \lim_{N \to \infty} \{N^{-1} \operatorname{Tr}[\exp(-\beta\kappa)]/\operatorname{Tr} \exp(-\beta\kappa)\} \tag{2}$$

denote the per spin ensemble average, where $\beta = 1/kT$ and the factor N^{-1} is to be deleted for the average of intensive quantities. Then

$$f(\kappa) = f(H_0) + kT\langle 1 - \exp[-\beta(\kappa - H_0)]\rangle_{H_0}^{c} \tag{3}$$

and

$$\langle \sigma_i \ldots \sigma_k \rangle_\kappa = \langle \exp[-\beta(\kappa - H_0)]\, \sigma_i \ldots \sigma_k \rangle_{H_0}^{c}. \tag{4}$$

Here $\langle\ \rangle_{H_0}^{c}$ denotes a 'connected' ensemble average defined by equation (2). In a connected ensemble average the factor $\exp[-\beta(\kappa - H_0)]$ in the numerator is expanded and only the connected graphs are kept in the expansion.

Corollary. $f(\kappa) = f(\kappa_1) + kT\langle \exp[-\beta(\kappa_1 - H_0)] - \exp[-\beta(\kappa - H_0)]\rangle_{H_0}^{c}$. (5)

Here κ_1 is any other Hamiltonian on the same spin system.

The MF model is an eight-vertex model (Lieb and Wu 1972) with vertex energies (Brascamp *et al* 1973)

$$\begin{aligned} e_1 &= e_2 = \epsilon_1 > 0, & e_3 &= e_4 = \epsilon_2 > 0. \\ e_5 &= e_6 = \mp s, & e_7 &= e_8 = \epsilon_1 + \epsilon_2 \end{aligned} \tag{6}$$

where s is the external (staggered) electric field; the $\mp$ signs refer respectively to vertices belonging to the two sublattices. This vertex model (6) is equivalent to an Ising model with first-neighbour interactions $-\frac{1}{2}s$ and second-neighbour interactions $\frac{1}{2}\epsilon_1$ and $\frac{1}{2}\epsilon_2$ respectively along the two diagonal directions (Brascamp *et al* 1973, Lieb and Wu 1972). We remark here that our notation conforms to that of Barber and Baxter (1973) and Baxter and Kelland (1974), provided that we write $s = 2E$.

Denote the spins on the two sublattices by σ_i and τ_l respectively. We can then write the Hamiltonian for the MF model as

$$\kappa(s) = \sum_\alpha \tfrac{1}{2}\epsilon_\alpha \left[\sum_{(i,j)}^{(\alpha)} \sigma_i\sigma_j + \sum_{(l,m)}^{(\alpha)} \tau_l\tau_m \right] - \tfrac{1}{2}s \sum_{(i,l)} \sigma_i\tau_l. \tag{7}$$

Here all summations are over neighbouring pairs, with $\alpha(=1, 2)$ denoting the diagonal directions. The staggered polarisation of the MF model is then (Brascamp *et al* 1973)

$$P(s) = -(\partial/\partial s)\, f(\kappa(s)). \tag{8}$$

The crux of our analysis is to use equation (5) to relate $f(\kappa(s))$ to an $f(\kappa_1(s))$ for which some information is known. Our choice is

$$\kappa_1(s) = \kappa_1(h) + \tfrac{1}{2}s \sum_{(i,l)} [m(h)]^2 \tag{9}$$

where

$$\kappa_1(h) = \kappa(0) - h\left(\sum_i \sigma_i + \sum_l \tau_l\right) \tag{10}$$

is the Hamiltonian of a nearest-neighbour Ising model in an external magnetic field h, and

$$m(h) = -(\partial/\partial h)\, f[\kappa_I(h)] = \langle\sigma_i\rangle_{\kappa_I(h)} = \langle\tau_l\rangle_{\kappa_I(h)} \tag{11}$$

is its magnetisation per spin.

For a given s, we choose h to satisfy

$$h = 2sm(h). \tag{12}$$

Substituting equations (9) and (12) into equation (5), we obtain

$$f(\kappa(s)) = f(\kappa_I(h)) + \tfrac{1}{2}hm(h) + kT\langle\exp\{-\beta[\kappa_1(s) - H_0]\} - \exp\{-\beta[\kappa(s) - H_0]\}\rangle^{c}_{H_0}. \tag{13}$$

It is readily verified that

$$\kappa(s) = \kappa_1(s) - \tfrac{1}{2}sH' \tag{14}$$

where

$$H' = \sum_{(i,l)} [\sigma_i - m(h)]\,[\tau_l - m(h)]. \tag{15}$$

Therefore, for small s, we may replace the last exponential factor in equation (13) by

$$\exp\{-\beta[\kappa(s) - H_0]\} = \exp\{-\beta[\kappa_1(s) - H_0]\}\,[1 - \tfrac{1}{2}sH' + O(s^2)]. \tag{16}$$

Further, using equation (4) and the fact that the spins σ_i and τ_l are decoupled in $\kappa_1(s)$, we have

$$\langle\exp\{-\beta[\kappa_1(s) - H_0]\}\,H'\rangle^{c}_{H_0} = \langle H'\rangle_{\kappa_1(s)} = 0. \tag{17}$$

This leads to our main result

$$f(\kappa(s)) = f(\kappa_I(h)) + \tfrac{1}{2}hm(h) + O(s^2). \tag{18}$$

Equation (18) shows that the MF model in a small electric field s is equivalent to a nearest-neighbour Ising model with an external magnetic field h, where s and h are related by equation (12).

It is now possible to compute directly the polarisation $P(s)$ of the MF model from the definition (8). Using equations (11) and (12), we obtain

$$P(s) = [m(h)]^2 + O(s) \tag{19}$$

which says that, to the order of s, the polarisation of the MF model is equal to the square of the magnetisation of a nearest-neighbour Ising model. For $s = h = 0$, equation (19) becomes exact and reads

$$P(0) = [m(0)]^2. \tag{20}$$

This expression relates the spontaneous polarisation of the MF model to the spontaneous magnetisation of the Ising model (Baxter and Kelland 1974). At the critical temperature T_c, writing the critical isotherms as

$$P(s) = A_e\, s^{1/\delta_e}$$
$$m(h) = A_m\, h^{1/\delta_m} \tag{21}$$

equation (19) leads to the relations

$$\delta_e = \tfrac{1}{2}(\delta_m - 1) \tag{22}$$

$$A_e = 2^{1/\delta_e} A_m^{2+1/\delta_e}. \tag{23}$$

Using the accepted value of $\delta_m = 15$ for the two-dimensional Ising model, we then obtain

$$\delta_e = 7. \tag{24}$$

This value confirms the prediction of the scaling and generalised homogeneous function hypotheses (Enting and Gaunt 1974). Equation (23) is new and relates the amplitude of the critical isotherm to that of the nearest-neighbour Ising model.

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