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LETTER TO THE EDITOR

Critical point of planar Potts models†

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Abstract. The critical point of the Potts model is conjectured for the following planar lattices: (i) generalised (chequerboard) square lattice; (ii) Kagomé lattice; (iii) triangular lattice with two- and three-site interactions. As a result, the critical probability for bond percolation on the Kagomé lattice is determined to be $p_c = 0{\cdot}524430$.

Properties of the Potts (1952) model are much more difficult to deduce than those of the Ising model. While the exact solution of the Ising model is now known for all planar lattices (for a list of the solutions for various Ising lattices, see Domb 1960 and Syozi 1972), the critical properties of the Potts model are only partially known, and are confined to specific lattices (Baxter 1973, Baxter *et al* 1978). In particular, the critical temperature has been established for the square, triangular and honeycomb lattices only (Hintermann *et al* 1978).

For the square lattice, the Potts critical point can be determined in a straightforward way from a duality argument (Potts 1952), while for the triangular and honeycomb lattices the argument is more complicated and involves additional steps (Kim and Joseph 1974, Baxter *et al* 1978, Hintermann *et al* 1978). It turns out that none of these analyses can be extended to other lattices.

For the purpose of providing useful reference points as well as for completing the list, it is desirable to have a knowledge of the Potts critical point for other lattices. We consider this problem in this Letter and make several conjectures. The conjectures are based on established results and plausible arguments, and are shown to be correct in various limits.

We shall consider q-component Potts models with ferromagnetic interactions. To facilitate discussions, we first summarise the existing known results.

The established critical point for the nearest-neighbour (ferromagnetic) Potts model on the square, triangular and honeycomb lattices are (Hintermann *et al* 1978)

$$x_1 x_2 = 1 \qquad \text{(square)} \tag{1}$$

$$\sqrt{q}\,x_1 x_2 x_3 + x_1 x_2 + x_2 x_3 + x_3 x_1 = 1 \qquad \text{(triangular)} \tag{2}$$

$$\sqrt{q} + x_1 + x_2 + x_3 = x_1 x_2 x_3 \qquad \text{(honeycomb)} \tag{3}$$

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where $x_r = [\exp(K_r) - 1]/\sqrt{q}$, $K_r = \epsilon_r/kT$, $\epsilon_r \geq 0$ being the interaction in the spatial direction $r = 1, 2, 3$. Note that the expressions (2) and (3) are related by the duality relation (Wu 1977)

$$x_r x_r^* = 1. \tag{4}$$

The critical condition for the triangular lattice has recently been extended (Baxter *et al* 1978, Wu and Lin 1979) to include three-site interactions $-\epsilon\delta_{ij}\delta_{jk}$ among the sites i, j, k surrounding every *other* triangular face. Here $\delta_{ij} = 1$ if the sites i and j are in the same state and $\delta_{ij} = 0$ otherwise. On the basis of a duality argument, the critical point of this model is located at

$$\exp(K_1 + K_2 + K_3 + K) = \exp(K_1) + \exp(K_2) + \exp(K_3) + q - 2 \tag{5}$$

where $K = \epsilon/kT$. For $K = 0$, equation (5) reduces to equation (2), as it should.

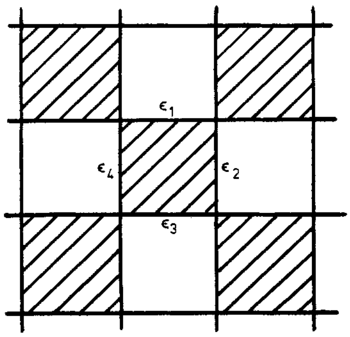

Figure 1. Generalised (chequerboard) square lattice. Each shaded square is bordered by interactions ϵ_1, ϵ_2, ϵ_3 and ϵ_4.

Generalised (chequerboard) square lattice. Consider the generalised (chequerboard) square lattice shown in figure 1. This lattice reduces to the honeycomb and triangular lattices respectively if one of the interactions is taken to be 0 or ∞. Thus its critical condition should reflect the same limits. Furthermore, since the generalised square lattice is self-dual, it follows from equation (4) that the criticality is invariant under the transformation $x_r \to x_r^{-1}$. These considerations then suggest the following expression for its critical point:

$$\sqrt{q} + x_1 + x_2 + x_3 + x_4 = x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 + \sqrt{q}x_1x_2x_3x_4. \tag{6}$$

Indeed, this is the only expression which is self-dual and which reduces to equations (3) and (2) upon taking $x_4 = 0$ and ∞ respectively. We conjecture that equation (6) is the correct critical condition.

The conjecture (6) is verified for $q = 2$ (the Ising model). In this case the exact critical point is known (Utiyama 1951, Domb 1960, Syozi 1972) to be

$$gdK_1 + gdK_2 + gdK_3 + gdK_4 = \pi \tag{7}$$

where $g\mathrm{d}K = 2\tan^{-1}\exp(K) - \pi/2$. It may be verified that equation (6) indeed reduces to equation (7) upon taking $q = 2$.

Triangular lattice with 2- and 3-site interactions. Consider next the triangular Potts model with the Hamiltonian

$$\mathscr{H} = -\sum_r \epsilon_r \sum_{ij} \delta_{ij} - \epsilon \sum_{\Delta} \delta_{ij}\delta_{jk} - \epsilon' \sum_{\nabla} \delta_{ij}\delta_{jk}. \tag{8}$$

Here, in addition to the two-site interactions ϵ_r, there are three-site interactions ϵ (ϵ') around each up-pointing (down-pointing) triangular face. This model has been studied by the renormalisation group technique (Schick and Griffiths 1977).

By symmetry we expect the critical condition of this model to be symmetric in ϵ_1, ϵ_2, ϵ_3 and also in ϵ and ϵ'. Now for $\epsilon' = 0$ the critical condition (for ferromagnetic interactions) is given by equation (5). The logical generalisation to $K' > 0$ is then

$$\exp(K_1 + K_2 + K_3 + K + K') = \exp(K_1) + \exp(K_2) + \exp(K_3) + q - 2. \tag{9}$$

We conjecture that equation (9) gives the critical point for the Potts model (8) for ferromagnetic interactions.

The conjecture (9) is again verified for $q = 2$. In this case the three-site interactions are reducible to two-site interactions (see e.g. Wu and Lin 1979). It is readily seen that, for $q = 2$, equation (9) agrees with the Ising exact result.

Kagomé lattice. Properties of the Potts model on the Kagomé lattice (figure 2) appear to be very elusive, and nothing is known at present. We shall, however, deduce its critical point from the conjecture (9).

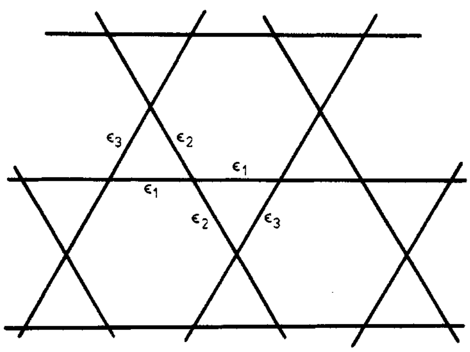

Figure 2. The Kagomé lattice with anisotropic interactions.

Consider first $\epsilon = \epsilon'$ in the Hamiltonian (8) and carry out a star–triangle transformation over every triangular face. This leads to the diced lattice as shown in figure 3. The transformation is well defined (Kim and Joseph 1974). Specifically, split each two-site interaction into two halves, each belonging to a neighbouring triangular face. As shown

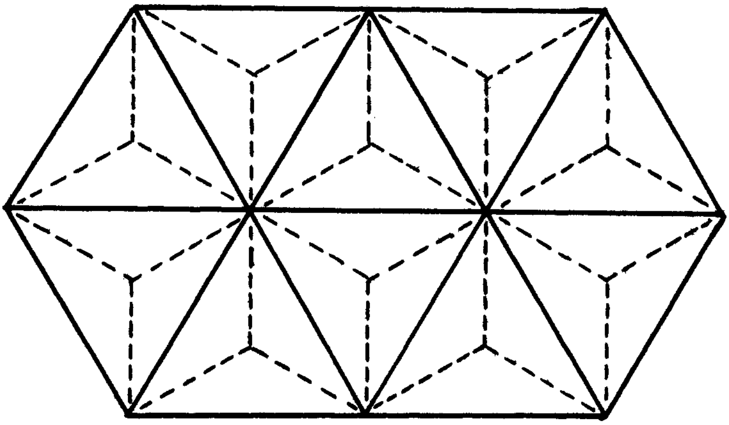

Figure 3. Star–triangle transformation relating the triangular (solid lines and the diced (broken lines) lattices.

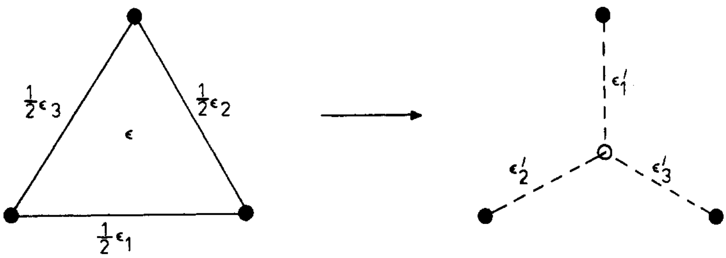

Figure 4. Details of the star–triangle transformation. ϵ is the three-site interaction.

in figure 4, the transformation reads

$$\begin{aligned}
&A \exp[\tfrac{1}{2}(K_1 + K_2 + K_3) + K] = \exp(K'_1 + K'_2 + K'_3) + q - 1 \\
&A \exp(\tfrac{1}{2}K_1) = \exp(K'_2 + K'_3) + \exp(K'_1) + q - 2 \\
&A \exp(\tfrac{1}{2}K_2) = \exp(K'_3 + K'_1) + \exp(K'_2) + q - 2 \\
&A \exp(\tfrac{1}{2}K_3) = \exp(K'_1 + K'_2) + \exp(K'_3) + q - 2 \\
&A = \exp(K'_1) + \exp(K'_2) + \exp(K'_3) + q - 3
\end{aligned} \tag{10}$$

where $K'_r = \epsilon'_r/kT$. We can solve for A, K_1, K_2, K_3 and K from equation (10). Substituting the solution into equation (9) and putting $K' = K$, we obtain the critical condition for the diced lattice. Finally, since the Kagomé and the diced lattices are mutually dual, the critical condition for the Kagomé lattice is deduced by applying the duality relation (4). The algebra is straightforward and we give here only the final expression:

$$\begin{aligned}
&qx_1^2x_2^2x_3^2 + 2\sqrt{q}x_1x_2x_3(x_1x_2 + x_2x_3 + x_3x_1) + 2x_1x_2x_3(x_1 + x_2 + x_3) \\
&\quad + x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2 \\
&\quad = 2\sqrt{q}x_1x_2x_3 + 4(x_1x_2 + x_2x_3 + x_3x_1) \\
&\quad + 2\sqrt{q}(x_1 + x_2 + x_3) + q.
\end{aligned} \tag{11}$$

We conjecture that equation (11) gives the critical point of the Kagomé Potts model.

The conjectured critical condition (11) is verified in two instances. As can be readily seen, for $q = 2$ equation (11) agrees with the exact transition point of the Ising model on the Kagomé lattice (Kano and Naya 1953, Domb 1960, Syozi 1972), which reads

$$\cosh K_1 \cosh K_2 \cosh K_3 + \sinh K_1 \sinh K_2 \sinh K_3 = \cosh K_1 + \cosh K_2 + \cosh K_3. \qquad (12)$$

Also, if one of the interactions ϵ_r vanishes, one can take the partial trace over two-thirds of the spins, and the result is a square Potts lattice. It is easily seen that in this instance equation (11) again leads to the exact result (1).

We plot in figure 5 the dependence of the critical temperature on q for various isotropic planar Potts lattices. For the square lattice the critical parameter $\exp(K) = 1 + \sqrt{q}$ is linear in $\sqrt{q}$, while for others the dependence on $\sqrt{q}$ is seen to be close to linear. The plot also shows a strong dependence of $\exp(K)$ on the coordination number z.

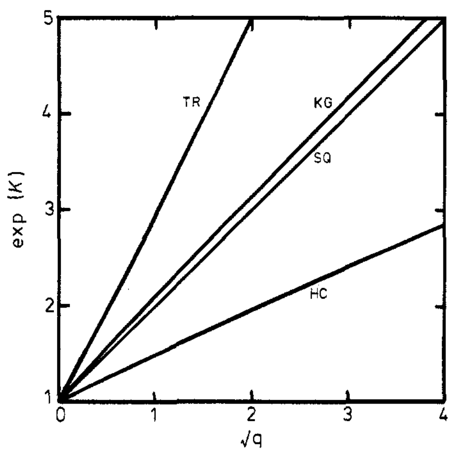

Figure 5. Critical parameter $\exp(K)$ as a function of $\sqrt{q}$ for the triangular (TR), Kagomé (KG), square (SQ) and honeycomb (HC) lattices.

For $q = 1$, the critical condition leads to the critical probability $p_c[=1 - \exp(-K)]$ of bond percolations (Kasteleyn and Fortuin 1969, Wu 1978). The result is listed in the following:

$$\begin{aligned} p_c &= 0{\cdot}652704 && (z = 3, \text{honeycomb}) \\ &= 0{\cdot}524430 && (z = 4, \text{Kagomé}) \\ &= 0{\cdot}5 && (z = 4, \text{square}) \\ &= 0{\cdot}347296 && (z = 6, \text{triangular}). \end{aligned} \qquad (13)$$

The value of p_c for the Kagomé lattice is new; other values in equation (13) have previously been obtained by Sykes and Essam (1964). The values of p_c indicate that the square lattice is somehow more 'close-packed' than the Kagomé lattice, although they both have the same coordination number $z = 4$.

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