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LETTER TO THE EDITOR

Phase diagram of a five-state spin system†

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Abstract. A classical five-state spin model, which includes both the standard and the vector Potts models as special cases, is considered. Two symmetry relations are derived and used to construct the phase diagram for this model. The result implies that the five-state vector Potts model has two phase transitions.

Consider a square net of N classical spins. The spins are confined in a plane and each spin can point along one of the five directions specified by the angle

$$\theta_k = 2\pi k/5 \qquad k = 0, 1, 2, 3, 4 \tag{1}$$

that the plane makes with, say, the x axis. The spins interact with a nearest-neighbour interaction which depends only on the relative angle between the two vectors. Thus, the Hamiltonian reads

$$\mathscr{H} = -\sum_{\langle ij\rangle} J(\theta_{ij}) \tag{2}$$

where $\theta_{ij} = \theta_{k_i} - \theta_{k_j}$ is the angle between the two spins at neighbouring sites i and j. The interaction $J(\theta)$ is assumed to be 2π-periodic. Since θ_{ij} can assume only five distinct values, we can write explicitly

$$J_k = J(\theta_k). \tag{3}$$

Symmetry now requires

$$J_1 = J_4 \qquad J_2 = J_3. \tag{4}$$

We shall further assume $J_0 \geqslant J_k$ to ensure ferromagnetism. This completes the description of the model. It is our goal to investigate the behaviour of the phase diagram of this model in the parameter space.

The spin model (2) is very general. By taking special values for J_k, for example, it reduces to the standard and the vector Potts model:

$$J_k = J \qquad k \neq 0 \qquad \text{(standard Potts)} \tag{5a}$$

$$J_k = J\cos\theta_k \qquad \text{(vector Potts)}. \tag{5b}$$

Other specialisations such as $\cos^2\theta$ interactions can be achieved in a similar manner.

The standard Potts model (Potts 1952) has been of interest for many years but the

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critical point of the five-state model is now known exactly (Hintermann *et al* 1978). In a recent paper Nishimori (1979) determined the critical point of the five-state vector Potts model on the basis of a 'pseudo' duality relation and the assumption of the existence of a unique transition. We shall now argue that the assumption of a unique transition is incorrect and, as a result, the critical point cannot be determined from the duality consideration. We also derive an exact duality relation for the spin model (2), which is a generalisation of the 'pseudo' duality described by Nishimori (1979).

Our consideration is based on two symmetry relations. Consider the partition function Z as a function of the Boltzmann factors

$$u_k = \exp(-J_k/k_B T) \tag{6}$$

and write $Z_{01234} \equiv Z(u_0, u_1, u_2, u_3, u_4)$. It is clear that a one-to-one correspondence exists between the spin configurations $\{\theta_{k_i}\}$ and $\{2\theta_{k_i}\}$. It then follows from the 2π-periodic property of $J(\theta)$ that

$$Z_{01234} = Z_{02413} = Z_{04321} = Z_{03142}. \tag{7}$$

Further, from equation (4) and introduction of the notation

$$\omega_1 \equiv u_1/u_0 \qquad \omega_2 \equiv u_2/u_0 \tag{8}$$

and

$$Z(\omega_1, \omega_2) \equiv Z(1, \omega_1, \omega_2, \omega_2, \omega_1) \tag{9}$$

we obtain from equation (7) the simple symmetry relation

$$Z(\omega_1, \omega_2) = Z(\omega_2, \omega_1). \tag{10}$$

We now make use of a result given by Wu and Wang (1976) to derive a second symmetry. Again, because the Boltzmann factors $u_k \equiv u(\theta_k)$ are 2π-periodic, the matrix $\mathbf{U}$ whose elements are $u_{kl} \equiv u(\theta_k - \theta_l)$ is cyclic. It then follows from the analysis of Wu and Wang (1976) that

$$Z(u_0, u_1, u_2, u_3, u_4) = 5^{-N} Z(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{11}$$

where

$$\lambda_l = \sum_{k=0}^{4} u_k \exp(2\pi i k l/5) \tag{12}$$

are the eigenvalues of the matrix $\mathbf{U}$. Here we have used the self-dual property of the square lattice explicitly and a periodic boundary condition is assumed.

Equation (11) is in fact a duality relation which generalises the 'pseudo' duality obtained by Nishimori (1979) for the vector Potts model. Note, however, that our derivation is more direct and transparent. The crux of the matter here is that although the vector Potts model is not self-dual, the duality is exact and rigorous for the more general spin model (2). With the help of equations (4) and (9), we now obtain the following from equation (11):

$$Z(\omega_1, \omega_2) = [(1 + 2\omega_1 + 2\omega_2)^2/5]^N Z(\omega_1', \omega_2') \tag{13}$$

where

$$\omega_1' = (1 + 2\omega_1 \cos\theta_1 + 2\omega_2 \cos\theta_2)/(1 + 2\omega_1 + 2\omega_2)$$
$$\omega_2' = (1 + 2\omega_1 \cos\theta_2 + 2\omega_2 \cos\theta_1)/(1 + 2\omega_1 + 2\omega_2). \tag{14}$$

Here $\cos\theta_1 = \cos(2\pi/5) = (\sqrt{5} - 1)/4$ and $\cos\theta_2 = \cos(4\pi/5) = -(\sqrt{5} + 1)/4$.

Equations (13) and (14) describe a transformation property of the partition function Z. It is readily verified that the transformation (14) maps the regions $\omega_1 + \omega_2 \gtrless (\sqrt{5} - 1)/2$ onto each other but leaves the line L: $\omega_1 + \omega_2 = (\sqrt{5} - 1)/2 = 0{\cdot}618$ invariant. The points (ω_1, ω_2) and (ω'_1, ω'_2) are colinear with the point $(\cos\theta_2, \cos\theta_2)$.

In light of the symmetry relations (10) and (13), we are now in a position to construct the phase diagram for the spin model (2). Here, the region of interest for a ferromagnetic model is $0 \leqslant (\omega_1, \omega_2) \leqslant 1$. First, the intersection of the phase boundary with the line $\omega_1 = \omega_2$ is known exactly. As seen from equation (5*a*), the line $\omega_1 = \omega_2$ describes the five-component standard Potts model. From the exact result given by Hintermann *et al* (1978), we know that along this line, labelled SP in figure 1, only one critical point exists;

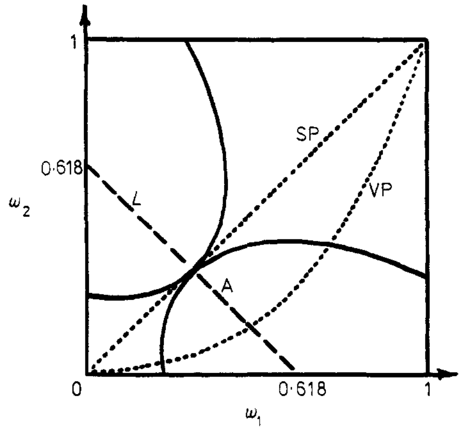

Figure 1. Schematic phase diagram for the spin model (2). Line L is self-dual under the transformation (13); SP denotes the thermodynamic path Γ of the standard Potts model and VP that of the vector Potts model.

this is the point A at $\omega_1 = \omega_2 = (\sqrt{5} - 1)/4$. This point also lies on the self-dual trajectory L so that, as expected, the symmetry relation (13) merely states the self-dual property of the five-state standard Potts model.

To acquire a complete picture of the phase diagram, we consider the possible orderings that will occur in different phases of the spin model. For small ω_1 and ω_2 we expect the system to be in a 'low-temperature' phase in which neighbouring spins tend to align in the same state, say θ_0. In the region near $(\omega_1, \omega_2) = (1, 0)\ (=(0, 1))$, we expect the interaction $J_1(J_2)$ to dominate resulting in an ordered phase in which neighbouring spins are in states θ_0 and $\theta_1(\theta_2)$. We also expect to find a disordered 'high-temperature' phase near $(\omega_1, \omega_2) = (1, 1)$. Thus, the four points (0, 0), (1, 0), (0, 1) and (1, 1) in the ω space are expected to lie in different regions separated by well-defined phase boundaries. This consideration, together with the relations (10) and (13) which reflect the full symmetry of the model, lead to the phase diagram shown schematically in figure 1. Note in particular that our consideration rules out the possibility that the self-dual line L may be a phase boundary and also that when similar reasonings are applied to the Ashkin–Teller model (Ashkin and Teller 1943), the phase diagram suggested by Wu and Lin (1974) is obtained.

It is now possible to discuss the occurrence of phase transition(s) in specific spin models. Generally, in a given spin model with fixed interactions, the variables (ω_1, ω_2) trace a certain thermodynamic path Γ in the ω space as the temperature in the physical model rises from 0 to ∞. A phase transition occurs whenever the path Γ intersects the phase boundary.

For the standard Potts model (equation 5*a*), Γ is the dotted straight line SP in figure 1. We have already seen that this line intersects with the phase boundary at only one point. This leads to a unique critical point, in agreement with the exact result (Hintermann *et al* 1978). For the vector Potts model (5*b*), Γ traces along the trajectory

$$\omega_1^{\sqrt{5}+1} = \omega_2^{\sqrt{5}-1} \tag{15}$$

labeled VP in figure 1. We see that Γ now intersects the phase boundary at two distinct points, leading to two phase transitions. Thus, the duality relation (13) merely relates the two critical points and does not locate the transition points.

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References

Ashkin J and Teller E 1943 *Phys. Rev.* **64** 178–84
Hintermann A, Kunz H and Wu F Y 1978 *J. Stat. Phys.* **19** 623–32
Nishimori H 1979 *Physica* A *preprint*
Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106–9
Wu F Y and Lin Y 1974 *J. Phys. C: Solid St. Phys.* **7** L181–4
Wu F Y and Wang Y K 1976 *J. Math. Phys.* **17** 439–40