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LETTER TO THE EDITOR

On the Temperley–Nagle identity for graph embeddings

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Abstract. A simple derivation of the Temperley–Nagle identity for graph embeddings is given. It is shown that the identity leads to a sum-rule relation connecting the lattice constants of strong and weak embeddings. It is also shown that the identity yields the fluctuation of the number of bonds (sites) in a site (bond) percolation.

The theory of graph embeddings plays an important role in the study of cooperative phenomena and related problems (Domb 1960, 1974). Two types of graph embeddings and, consequently, two types of lattice constants arise in these considerations. These are the lattice constants associated with the strong (low-temperature) embeddings and those with the weak (high-temperature) embeddings. It has been shown by Sykes *et al* (1966) that the two sets of lattice constants are related by a relationship which is most easily derived from a geometric consideration. To our knowledge, this result has remained to this date the only explicit relation which connects these two sets of lattice constants.

Some ten years ago Nagle (1968) obtained a weak-graph expansion for the low-temperature Ising problem to arrive at an identity which has also been described earlier by Temperley (1959). One interesting consequence of this expansion, which was not made explicit in Nagle (1968), is a sum-rule relation connecting the weak and strong lattice constants. Another implication of this Temperley–Nagle (TN) identity is that it yields the fluctuation of the number of bonds (sites) in a site (bond) percolation. We describe these results in this Letter. We also give an alternate but more elegant derivation of the TN relation.

Consider an arbitrary graph G of N vertices (sites) and E lines (edges) which in the case of physical applications will be a lattice. Let $G^+ \subseteq G$ be a section graph of G of v vertices and l lines, and L a line set of G containing l lines which cover v vertices. (A vertex is considered covered if its degree is at least one.) The TN identity now reads

$$\sum_{G^+} x^v y^l = (1+x)^N \sum_L (y-1)^l \left(\frac{x}{1+x}\right)^v \qquad (1)$$

where the summation on the left-hand (right-hand) side of (1) extends over all section graphs (line sets) of G. To facilitate discussions we first discuss the implications of (1). A simple derivation of this identity will be given at the end of the Letter.

The left-hand side of (1) generates precisely the strong embeddings of all section graphs of G. The right-hand side of (1), which extends over the line sets L, generates the

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weak embeddings of the subgraphs of G that contain no isolated sites. Thus, if we define

$[v, l; G]$ = the number of strong embeddings in G of all section graphs $G^+ \subseteq G$ of v vertices and l lines,

$(v, l; G)$ = the number of weak embeddings in G of all subgraphs $G' \subseteq G$ of v vertices and l lines.

and adopt the convention that $(v, l; G) \equiv 0$ whenever the subgraph G' contains isolated sites, then (1) can be rewritten as

$$\sum_{v,l} [v, l; G]x^v y^l = \sum_{v,l} (v, l; G)(y-1)^l x^v (1+x)^{N-v}. \tag{2}$$

By expanding the right-hand side and equating the coefficients of $x^v y^l$, we obtain the identity

$$[\bar{v}, \bar{l}; G] = \sum_{v=0}^{\bar{v}} \sum_{l=\bar{l}}^{\frac{1}{2}v(v-1)} (-1)^{l-\bar{l}} \binom{N-v}{\bar{v}-v} \binom{l}{\bar{l}} (v, l; G) \tag{3}$$

where

$$\binom{n}{r} = n(n-1)\ldots(n-r+1)/r!.$$

Similarly we obtain the inverse relation

$$(\bar{v}, \bar{l}; G) = \sum_{v=0}^{\bar{v}} \sum_{l=\bar{l}}^{\frac{1}{2}v(v-1)} (-1)^{v-\bar{v}} \binom{N-v}{\bar{v}-v} \binom{l}{\bar{l}} [v, l; G]. \tag{4}$$

The expressions (3) and (4) relate the sums of strong and weak lattice constants of fixed numbers of vertices and lines. Therefore they are in fact sum rules for the constants of individual graphs. Note that these relations do not distinguish between connected and disconnected constants. Also they are valid for general N and easy to write down, for its coefficients are simple combinatorial factors that require no geometric considerations. The correctness of (3) and (4) for specific values of $\bar{v}$ and $\bar{l}$ can be readily verified. For example, for $\bar{v} = 3$ and $\bar{l} = 1$ one obtains the identity

$$[3, 1; G] = 3(3, 3; G) - 2(3, 2; G) + (N-2)(2, 1; G).$$

Thus, using the following weak lattice constants:

$$(3, 3; G) = 0, 2N, 8N$$

$$(3, 2; G) = 6N, 15N, 66N$$

$$(2, 1; G) = 2N, 3N, 6N$$

respectively for a periodic square, triangular, or FCC lattice of N sites, we generate the respective strong lattice constants

$$[3, 1; G] = 2N(N-8), 3N(N-10), 6N(N-20).$$

For $N = 1$, these expressions yield the numbers listed in Domb (1960).

As another implication of the identity (1) we observe that by taking $x = p/(1-p)$ and $y = 1$ the left-hand side of (1) generates a site percolation process on G in which each site has a probability p to be occupied and a probability $1-p$ to be empty. Similarly, by taking $y-1=p/(1-p)$ and $x=\infty$ the right-hand side of (1) generates a bond percolation process on G. This observation permits the explicit evaluation of the mean values of the *global* moments $\langle v^m l^n\rangle$ for both the site and bond percolations. For example, it is clear that the average number of occupied bonds in a site percolation is

$$\langle l\rangle_s = Ep^2 \tag{5}$$

and the average number of occupied sites in a bond percolation is

$$\langle v\rangle_b = N[1-(1-p)^z] \tag{6}$$

where z is the coordination number of the lattice. The identity (1) permits the further evaluation of the fluctuations from the mean. For example, the quantity

$$\langle l(l-1)\rangle_s = \left[\frac{\partial^2}{\partial y^2}\sum_{G^+} x^v y^l\right]_{y=1}$$

is most conveniently evaluated by taking the derivatives on the right-hand side of (1) at $y=1$. It is then straightforward to deduce the following expressions for the fluctuations:

$$\langle l^2\rangle_s - \langle l\rangle_s^2 = N[p^2+2(z-1)p^3-(2z-1)p^4] \tag{7}$$

$$\langle v^2\rangle_b - \langle v\rangle_b^2 = E[(1-p)^z p(1-p)^{2z-1}]. \tag{8}$$

Finally we give a simple derivation of the TN identity (1). Nagle (1968) arrived at (1) by applying the weak-graph expansion method to the Ising low-temperature expansion. A simpler and more elegant derivation is to consider the following process on G:

(i) each vertex of G is either occupied, e.g. covered by an atom, or empty;
(ii) each line of G is either occupied, e.g. covered by a bond, or empty; and
(iii) a line can be occupied only when the two vertices it connects are occupied.

If a vertex is occupied, it carries a weight x, and each occupied line carries a weight u. The generating function for the process (i)–(iii) is then

$$\Xi(x,u) = \sum_V \sum_L{}' x^v u^l \tag{9}$$

where the summations extend overall occupational configurations V and L of the vertex and line sets of G. The prime denotes the imposition of the restriction (iii), and v and l are the numbers of the occupied vertices and lines.

Next we carry out the summations in (9) in two different orders. First, we sum over all allowed line occupations for a given vertex occupation. Now the vertex occupations are most conveniently designated by the section graphs $G^+\subseteq G$ whose vertex set contains precisely the occupied sites. Also, according to (iii), each line in G^+ can be either occupied or empty. We therefore arrive at

$$\Xi(x,u) = \sum_{G^+} x^v(1+u)^l \tag{10}$$

where v and l are the numbers of vertices and lines in G^+. Similarly, we may sum over all allowed vertex occupations for a given line occupation in (9). The line occupations are most conveniently designated by the line set L of the occupied edges. Also according to (iii) each isolated site can be either occupied or empty, we then obtain

$$\Xi(x, u) = \sum_{L} u^l x^v (1+x)^{N-v} \tag{11}$$

where v and l are the numbers of covered vertices and lines in L. The identity (1) now follows from (10) and (11) upon putting $y = 1 + u$.

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