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Absence of phase transitions in treelike percolation in two dimensions

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It is shown on the basis of a continuity argument that the treelike percolation on the square, triangular, and honeycomb lattices does not have a phase transition.

In a recent paper, Stephen[1] proposed the problem of a treelike percolation on a lattice and suggested the occurrence of a phase transition in such systems. Indeed, the existence of such a transition is not without interest. Recent studies based on both Monte Carlo data[2,3] and exact analyses[4,5] indicate that in the ordinary percolation the shapes of large clusters near the percolation limit are highly ramified (treelike). In fact, in one soluble model of bond percolation on a *complete* graph,[6] essentially all clusters are of the tree type below the percolation threshold. It is then pertinent to ask whether the percolation threshold would persist in an ensemble in which only the treelike clusters are included. This leads to the treelike percolation. While the usual questions of a percolation can be asked for this problem,[1,7] the purpose of this paper is to point out that the treelike percolation does not have a phase transition in two dimensions.

We consider a treelike bond percolation in which the edges of a lattice $\mathcal{L}$ are "occupied" independently by bonds with a probability p, with the occupied edges forming only trees, which are subgraphs of $\mathcal{L}$ containing no cycles or loops. While the treelike percolation can be formulated[1] as the infinite-temperature zero-component limit of the Potts model,[8] it is more convenient to use the Whitney polynomial[9,10]

$$W(x,y)=\sum_G x^{e-c}y^c \tag{1}$$

as the generating function for the problem. The summation in Eq. (1) extends to *all* subgraphs $G\subseteq\mathcal{L}$ of $e\equiv e(G)$ edges and $c\equiv c(G)$ independent cycles. It is clear that only the tree graphs ($c=0$) survive if we set $y=0$ in Eq. (1). Thus

$$W(x,0)=\sum_{\text{trees}} x^e \tag{2}$$

generates the tree configurations in the treelike percolation provided that we take

$$x=p/(1-p)\,. \tag{3}$$

The connection between the Whitney polynomial and the treelike percolation can be taken one step further by considering the mean number of trees per site $n(p)$. For a large lattice of N sites we define the Whitney function as

$$w(x,y)=\lim_{N\to\infty}N^{-1}\ln W(x,y)\,. \tag{4}$$

It follows that

$$n(p)=1-x\frac{d}{dx}w(x,0)\,, \tag{5}$$

where p and x are related by Eq. (3) and use has been made of the fact that the number of trees in a tree configuration is simply $N-e$. This formulation is valid for lattice $\mathcal{L}$ in any dimension.

If a transition occurs in the treelike percolation, we expect as in the usual bond or site percolation[11] that $n(p)$ becomes nonanalytic in p at some $0<p_c<1$, where p_c is the critical probability. It is clear from Eq. (5) that the singularity of $n(p)$, if any, coincides with that of $w(x,0)$. In order to locate the singularity of the function $w(x,0)$, we now consider more generally the Whitney function $w(x,y)$. The ensuing discussion is confined to $\mathcal{L}$ in two dimensions.

For planar lattices the Whitney function is equivalent to the free energy of an ice-type vertex model.[12] From this equivalence and the established results on the vertex models,[5,13] we know that $w(x,y)$ is nonanalytic in the region $x,y>0$ at

$$\begin{aligned}&xy=1\,,\quad \text{square lattice}\,,\\&x(3y+y^2)=1\,,\quad \text{triangular lattice}\,,\\&1+3x=x^2y\,,\quad \text{honeycomb lattice}\,.\end{aligned} \tag{6}$$

In fact, it has been established rigorously[14] that $w(x,y)$ can be nonanalytic *only* at Eq. (6) in the region $y>4x>0$. Indeed, the Whitney function for a square lattice satisfies the duality relation[10]

$$w(x,y)=\ln(xy)+w(y^{-1},x^{-1})\,, \tag{7}$$

which maps a point (x,y) to (y^{-1},x^{-1}). If one makes the usual assumption[15] of a unique singular point along the "Potts" line $y/x=$ constant, then the singular point is always given by the self-dual-point $xy=1$. A similar argument can be made for the triangular and honeycomb lattices leading to a transition point which is the same as that given in Eq. (6) (Refs. 16 and 17). Thus it appears safe to conclude that $w(x, y)$ is an analytic function for all $x, y > 0$ except at Eq. (6).

To draw conclusions on the location of the singular point of $w(x, 0)$, we now invoke a continuity argument which assumes a continuous dependence of the criticality of a lattice statistical model on its parameters.[18] In the present problem, we assume a continuous dependence of the singularity of the Whitney function $w(x, y)$ on the arguments x and y. The singularity of $w(x, 0)$ can thus be determined from Eq. (6) by taking the limit $y \to 0$. Since in this limit the only solution of Eq. (6) is $x = \infty$ or $p = 1$, it follows that $n(p)$ is nonanalytic in p only at $p = 1$. Consequently, the treelike percolation does not have a phase transition, except at the trivial point $p = 1$. While this result is not in disagreement with the earlier finding of the dominance of the treelike clusters below the percolation threshold in an ordinary percolation, it does single out the important role of the non-tree-like clusters in bringing about a transition at $p_c < 1$.

Unfortunately, not much is known about the Whitney function for three-dimensional lattices. However, by slightly extending the above argument, it is possible to relate the existence of a transition in the treelike percolation with a certain condition on the criticality of the related Potts model. Using the equivalence of the Whitney polynomial and the Potts partition function,[12] which is valid in any dimension, it can be seen that the treelike percolation exhibits a transition only if the critical condition of the q-state Potts model takes the form

$$e^{\epsilon / kT} = 1 + qh(q) , \tag{8}$$

with the function $h(q)$ satisfying $0 < h(0) < \infty$, ϵ being the difference between the interactions of unlike and like Potts states. For two-dimensional lattices, Eq. (6) implies $h(0) = \infty$ and, as a consequence, there is no transition in the treelike percolation.

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