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LETTER TO THE EDITOR

Site percolation as a Potts model

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Abstract. The site percolation on a lattice is formulated as a Potts model with many-body interactions. Quantities of interest in the site percolation are expressed in terms of the Potts partition function in the same way as in the Kasteleyn–Fortuin formulation of the bond percolation.

An important development in the theory of percolation is the formulation of the bond problem as a Potts model (Kasteleyn and Fortuin 1969). This formulation permits a Hamiltonian approach to the bond percolation and has led to renewed interest in percolation problems in recent years. A reformulation of the Kasteleyn–Fortuin result in a more compact form has been given by Wu (1978) and by Stephen (1977).

In this Letter we extend the Kasteleyn–Fortuin result to the site problem. We show that the site percolation is also related to a Potts model in much the same way as the bond problem. In particular, quantities of interest arising in the percolation problem are given in terms of some derivatives of Potts partition function.

Consider a lattice G of N sites and coordination number z. The covering lattice G_c of G is defined with its $\frac{1}{2}zN$ sites located on the edges of G (assuming periodic boundary conditions). To consider the site percolation on G, we introduce a q-state Potts model on G_c with the following Hamiltonian:

$$\mathscr{H} = -\sum_{i=1}^{N} [J\delta(\{\xi_i\}) + H\delta_\alpha(\{\xi_i\})] \tag{1}$$

where $\{\xi_i\}$ denotes the set of the z Potts spin states of the sites surrounding the ith site of G,

$$\begin{aligned} \delta(\{\xi_i\}) &= 1, \quad \text{if all } z \text{ spins are in the same state} \\ &= 0, \quad \text{otherwise} \end{aligned} \tag{2}$$

and

$$\begin{aligned} \delta_\alpha(\{\xi_i\}) &= 1, \quad \text{if all } z \text{ spins are in the state } \alpha \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

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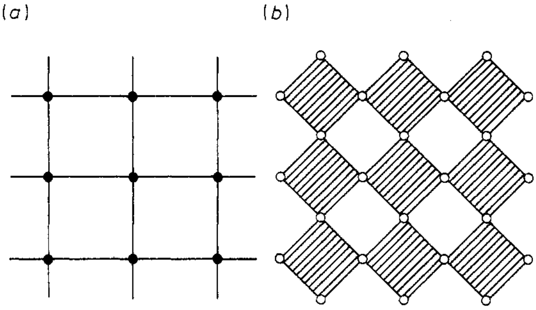

Figure 1. (*a*) The square lattice G for the site percolation.
(*b*) The covering lattice G_c of G. The shaded areas denote the four-body interactions and the external applied field in the Potts model (1).

Thus the Hamiltonian (2) describes a Potts model with a z-body interaction $-J$ and an external field $-H$ applied to state α. We illustrate in figure 1 the situation for a square lattice ($z = 4$), where the four-body interactions and the external field are denoted by the shaded regions.

The partition function of the Potts model (1) reads

$$Z(q; v, L) = \sum_{\xi_i=1}^{q} \prod_i [1 + v\delta(\{\xi_i\})] \prod_i [1 + u\delta_\alpha(\{\xi_i\})] \tag{3}$$

where

$$v = e^K - 1, \qquad u = e^L - 1,$$
$$K = J/kT, \qquad L = H/kT.$$

Expand the first product in (3) and use the subgraphs of G to represent the terms in the expansion. Each term in the expansion is conveniently represented by a subgraph $G' \subseteq G$ whose site set coincides with the v factors contained in the term. Further identify the sites in G' as being 'occupied' as in a site percolation, we then have a one-to-one mapping between the configurations of site percolation (on G) and the terms in the expansion of (3). A typical example of this mapping for the square lattice is shown in figure 2.

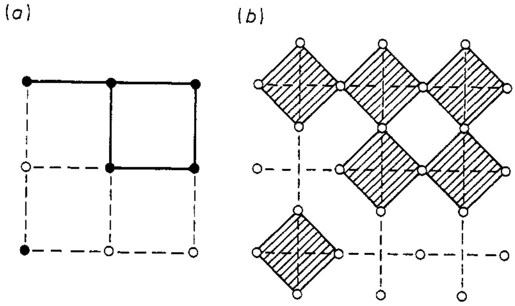

Figure 2. Mapping between the configurations of site percolation and the terms in the expansion of (3):
(*a*) A typical configuration of the site percolation. The black circles denote the occupied sites and the neighbouring occupied sites are connected by bonds.
(*b*) Graphical representation of the corresponding term in (3). Each shaded area denotes a factor v.

Note that the neighbouring occupied sites are connected by bonds so that the occupied sites form connected clusters.

Next carry out the spin sums in (3). The summations yield a factor q for each isolated site on G_c and a factor $e^{Ls_c} + q - 1$ for each connected piece on G_c containing s_c v's. To count the number of isolated sites on G_c we note that each occupied site on G reduces this number by z whereas each bond raises it by 1. Hence there are a total of $\frac{1}{2}zN - zs + b$ isolated sites on G_c, here b and s are respectively, the numbers of bonds and sites in G'. It follows then that (3) takes the form

$$Z(q; v, L) = q^{zN/2} \sum_{G'} v^s \prod_{c} [q^{b_c - zs_c}(e^{Ls_c} + q - 1)] \tag{4}$$

where the product is over all clusters in G' having b_c bonds and s_c sites.

Define the free energy per site for the Potts model

$$f(q; v, L) \equiv \lim_{N \to \infty} N^{-1} \ln Z(q; v, L) \tag{5}$$

and the function

$$h(p, L) \equiv \left[\frac{\partial}{\partial q} f(q; v\,e^{-L}, L)\right]_{q=1} \tag{6}$$

with

$$p = 1 - e^{-K}. \tag{7}$$

Then $h(p, L)$ is the generating function for the site percolation, p being the site occupation probability. Combining (4)–(7), we find

$$h(p, L) = \tfrac{1}{2}z + \langle b \rangle_0 - z\langle s \rangle_0 + \langle \sum_{c} e^{-Ls_c} \rangle_0 \tag{8}$$

where, as in Wu (1877) $\langle\rangle_0$ denotes the per site average. Quantities of interest arising in the site percolation can now be expressed in terms of the function $h(p, L)$. In particular, we find the percolation probability $P(p)$ and the mean square cluster size $S(p)$ given by

$$P(p) = 1 + p^{-1}\left[\frac{\partial}{\partial L} h(p, L)\right]_{L=0+} \tag{9}$$

$$S(p) = \left[\frac{\partial^2}{\partial L^2} h(p, L)\right]_{L=0+}. \tag{10}$$

These expressions are the same as those for the bond problem. However, expression for the 'free energy' for the site problem is now different and takes the form

$$h(p, 0) = \tfrac{1}{2}z + \langle b - zs + n \rangle_0 \tag{11}$$

where n is the number of clusters. The corresponding expression for the bond percolation is $\langle n \rangle_0$. The Griffiths inequality for critical exponents can also be established by using (8)–(10) as in Wu (1977).

Finally, the pair connectivity of the site problem is connected to the correlation function of the Potts model (1). Let $P_{\alpha\alpha}(r)$ denote the probability that the spins surrounding the sites at $\mathbf{0}$ and r on G are *all* in the state α. The pair connectivity $c(r, p)$, which is the probability that in a site percolation the sites at **0** and $\mathbf{r}$ are in the same cluster, is then given by

$$c(\mathbf{r}, p) = [\partial \Gamma_{\alpha\alpha}(\mathbf{r})/\partial q]_{q=1} \tag{12}$$

where $\Gamma_{\alpha\alpha} = P_{\alpha\alpha} - q^{-2}$. The derivation of (12) is the same as that for the bond problem (Wu 1978).

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