

LETTER TO THE EDITOR

Number of spanning trees on a lattice†

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Abstract. The number of spanning trees on a large lattice is evaluated exactly for the square, triangular and honeycomb lattices.

A spanning tree of a lattice \mathcal{L} is a graph drawn on \mathcal{L} which connects all lattice sites and contains no polygons. For a regular lattice of N sites, the number of spanning trees on \mathcal{L} , T_N , behaves as e^{zN} for large N . We report here the exact values of z for the square (SQ), triangular (TR) and honeycomb (HC) lattices.

More specifically let

$$z = \lim_{N \rightarrow \infty} N^{-1} \ln T_N. \tag{1}$$

We find

$$z_{\text{SQ}} = \frac{4}{\pi} (1 - 3^{-2} + 5^{-2} - 7^{-2} + \dots) = 1.166\ 243\ 6\dots$$

$$z_{\text{TR}} = \frac{3\sqrt{3}}{\pi} (1 - 5^{-2} + 7^{-2} - 11^{-2} + 13^{-2} - \dots) = 1.615\ 329\ 68\dots \tag{2}$$

$$z_{\text{HC}} = \frac{1}{2} z_{\text{TR}} = 0.807\ 664\ 8\dots$$

It was first pointed out by Fortuin and Kasteleyn (1972) that T_N is expressible in terms of the partition function of a lattice statistical model. For our purpose, it suffices to consider the following graph generating function on \mathcal{L} :

$$Z_N(q, v) = \sum_G q^n v^e. \tag{3}$$

Here the summation extends over all graphs G on \mathcal{L} ; n and e are, respectively, the numbers of clusters and edges in G . Z_N is proportional to the cluster generating function of Fortuin and Kasteleyn (1972), and coincides with the partition function of a q -component Potts model for integral q (Baxter 1973). Now let $v = q^\alpha$ and consider the $q \rightarrow 0$ limit of the function

$$Z_N(q, q^\alpha) = q^{\alpha N} \sum_G q^{\alpha c + (1-\alpha)n} \tag{4}$$

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where we have used the Euler relation $N+c = n+e$ to eliminate the parameter e in favour of c , the number of independent circuits in G . For $\alpha = 1$ the leading terms in (4) in the $q \rightarrow 0$ limit are the tree graphs ($c = 0$). Consequently, $Z_N(q, q)$ generates forests of trees on \mathcal{L} (Stephen 1976). For $0 < \alpha < 1$ the leading terms are the spanning trees ($c = 0, n = 1$). Thus we have the exact relation valid for any finite lattice

$$T_N = \lim_{q \rightarrow 0} q^{\alpha(1-N)-1} Z_N(q, q^\alpha), \quad 0 < \alpha < 1. \tag{5}$$

Equation (5) reduces to (7.13) of Fortuin and Kasteleyn (1972) upon taking $\alpha = \frac{1}{2}$.

For planar \mathcal{L} the generating function (3) is related to the partition function of an ice-type problem on a related medial lattice \mathcal{L}' (Baxter *et al* 1976). The choice of $\alpha = \frac{1}{2}$ in (5) is especially convenient, for the resulting ice-type model is well defined and soluble in the $q \rightarrow 0$ limit. Combining (5) with (14) of Baxter *et al* (1976), we obtain (with $\alpha = \frac{1}{2}$) from (1)

$$z = \lim_{N \rightarrow \infty} N^{-1} \ln Z' \tag{6}$$

where Z' is the partition function of an ice-type model defined on the medial lattice \mathcal{L}' . If \mathcal{L} is a square lattice, then \mathcal{L}' is also a square lattice but having $2N$ sites. The vertex weights of the ice-type model on \mathcal{L}' are (cf figure 5 of Baxter *et al* 1976)

$$\omega_1, \dots, \omega_6 = 1, 1, 1, 1, \sqrt{2}, \sqrt{2}. \tag{7}$$

If \mathcal{L} is a triangular (honeycomb) lattice, then \mathcal{L}' is a Kagomé lattice of $3N$ ($3N/2$) sites with the following vertex weights:

$$\omega_1, \dots, \omega_6 = 1, 1, 1, 1, e^{-i\pi/6} + e^{i\pi/3}, e^{i\pi/6} + e^{-i\pi/3}. \tag{8}$$

In either case, it is readily verified that the weights satisfy the free-fermion conditions (Fan and Wu 1970)

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 \tag{9}$$

so that the right-hand side of (6) can be evaluated by computing a Pfaffian. In the case of square lattice, the free-fermion solution of Z' was first evaluated by Wu and reported in Lieb (1967). The numerical value (2) for z_{SQ} now follows from (20) of Lieb (1967) and the fact that \mathcal{L}' contains $2N$ sites. In the case of Kagomé lattice, the free-fermion solution of Z' has been obtained by Lin (1975)†. In the notation shown in figure 2 of Lin (1975), we may rewrite (8) as

$$\begin{aligned} \omega_i &= \omega'_i = \omega''_i, & i &= 1, 2, 3, 4 \\ \omega_5 &= \omega'_5 = \omega''_6 = e^{-i\pi/6} + e^{i\pi/3} \\ \omega_6 &= \omega'_6 = \omega''_5 = e^{i\pi/6} + e^{-i\pi/3} \end{aligned} \tag{10}$$

Equation (11) of Lin (1975)† now leads to

$$z_{TR} = 2z_{HC} = \frac{1}{4\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln[6 - 2 \cos \theta - 2 \cos \phi - 2 \cos(\theta + \phi)]. \tag{11}$$

This reduces to (2) upon carrying out the integrations.

† The free energy given by Lin (1975) contains an error. The right-hand side of his equation (11) (and all other expressions for ϕ) should be multiplied by a factor $\frac{2}{3}$.

References

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