Ashkin–Teller model as a vertex problem*

F. Y. Wu

Department of Physics, Northeastern University, Boston, Massachusetts 02115
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It is shown that the Ashkin–Teller model on any planar lattice is equivalent to an eight-vertex model on a related lattice. The exact equivalence is given for finite lattices with a boundary. We show, in particular, that the AT model on the triangular or honeycomb lattice is related to an eight-vertex model on a Kagomé lattice. The occurrence of two phase transitions in the AT model in general is also discussed.

There has been considerable recent interest in the Ashkin–Teller (AT) model[1] and the Potts model[2] considered as models for phase transitions in lattice systems. Although neither of these models has been solved to this date, interesting transformation properties have been found which relate these models to other lattice statistical problems. For the Potts model on a square lattice, Temperley and Lieb[3] showed that the model is related to a six-vertex problem. This relationship has since been generalized to arbitrary planar lattice,[4] and has been used to obtain properties of the Potts model at the critical point.[5] For the AT model on a square lattice, it was shown by Wegner[6] that the model is reducible to an eight-vertex problem. It does not appear to be widely known, however, that the validity of this equivalence is more general, also extendible to arbitrary planar lattices. It is the purpose of this note to explicitly derive this relation. We shall establish the equivalence for finite lattices with a boundary. We also briefly discuss the occurrence of two phase transitions in the AT model in general when the lattice becomes infinite.

Consider a planar lattice L composed of N sites and E edges. For simplicity we shall assume L be multiply connected and without single-edge loops, but otherwise it is arbitrary. Let the sites of L be occupied by atoms which can be in one of four states, A, B, C, or D. Two atoms occupying the sites connected by an edge interact with an energy ϵ_0 if they are in the same state. The interaction is ϵ_1 between states AB, CD, ϵ_2 between AC, BD, and ϵ_3 between AD, BC. This defines the AT model. We shall suppose $\epsilon_0 \leq \epsilon_i$, $i=1, 2, 3$, so that the system is analogous to an Ising ferromagnet.

The Ising representation of the AT model introduced by Fan[7] proves to be useful. Introduce at the ith site of L two Ising spins (σ_i, τ_i) and identify the spin configuration $(+,+)$ as state A, $(+,-)$ as B, $(-,+)$ as C, and $(-,-)$ as D. Then, the energies ϵ_0, ϵ_1, ϵ_2, ϵ_3 can be represented by 2 and 4 spin interactions between neighboring pairs of spins (σ, τ) and (σ', τ'), and the partition function of the AT model takes the form

$$Z_{\mathrm{AT}}(\omega_0, \omega_1, \omega_2, \omega_3) = \sum_{\{\sigma,\tau\}} \prod_{\{E\}} \exp(K_0 + K_1\sigma\sigma' + K_2\tau\tau' + K_3\sigma\sigma'\tau\tau'), \quad (1)$$

where

$$\omega_i = \exp(-\epsilon_i/kT),$$
$$\exp(4K_0) = \omega_0\omega_1\omega_2\omega_3, \qquad \exp(4K_1) = \omega_0\omega_1/\omega_2\omega_3, \quad (2)$$
$$\exp(4K_2) = \omega_0\omega_2/\omega_1\omega_3, \qquad \exp(4K_3) = \omega_0\omega_3/\omega_1\omega_2.$$

The product in (1) is taken over the edge set $\{E\}$ of L, and Z_{AT} is symmetric in $(\omega_1, \omega_2, \omega_3)$ or (K_1, K_2, K_3).

Following Wegner,[6] we now effect a duality transformation[8,9] on one set of the spins, say $\{\tau\}$. For this purpose the sites of L_D, the dual of L, are located inside the faces of L, and the area outside the boundary of L is considered as one single face.[10] The number of of sites N_D of L_D is then given by the topological relation

$$N_D = E - N + 2. \quad (3)$$

Let μ_i be the Ising spin at the ith site of L_D. The exact duality relation for a finite Ising model now reads[8,9]

$$\sum_{\{\tau\}} \prod_{\{E\}} \exp(K_{ij}\tau_i\tau_j) = 2^{1-N_D} \sum_{\{\mu\}} \prod_{\{D\}} [\exp(K_{ij}) + \mu_i\mu_j \exp(-K_{ij})], \quad (4)$$

where the product on the rhs of (4) is taken over the edge set $\{D\}$ of L_D. We have also assumed the Ising parameters in (4) to be edge-dependent.

It is now a simple matter to substitute (4) into (1) to obtain

$$Z_{\mathrm{AT}} = 2^{1-N_D} \sum_{\{\sigma,\tau\}} \prod_{\{E\}} w(\sigma\sigma', \mu\mu'), \quad (5)$$

where the factors

$$w(\sigma\sigma', \mu\mu') = \exp(K_0 + K_1\sigma\sigma')\{\exp(K_2 + K_3\sigma\sigma') + \mu\mu' \exp[-(K_2 + K_3\sigma\sigma')]\} \quad (6)$$

are associated with the edges. It is convenient at this point to introduce the medial lattice[4,11] M obtained by connecting the midpoints of the adjacent edges of L. An example is shown in Fig. 1. The sites of M are located on the edges of L (or L_D) and the edges of M separate

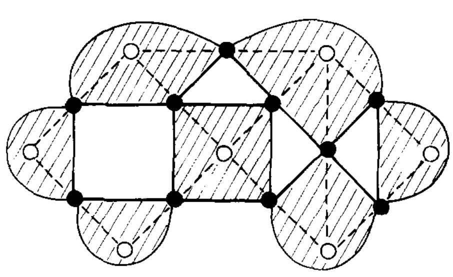

FIG. 1. A lattice L (open circles and broken lines) and its medial lattice M (solid circles and full lines). The 7 σ spins at the open circles are located inside the shaded faces of M and the 5 μ spins inside the plain faces. The number of sites on M is $E=10$.

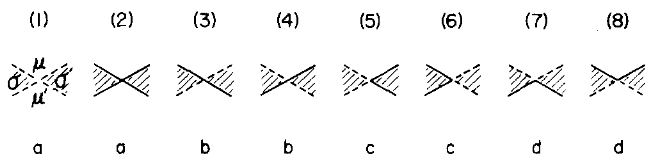

FIG. 2. The 8 vertex configurations on M and the associated vertex weights.

the σ spins from the μ spins. The faces of M containing the σ spins have been shaded in Fig. 1. Since we allow curved edges in M, it is not necessary to introduce the "exterior" sites of M as in Refs. 4 and 11, and the medial lattice has the exact coordination number 4 at all sites.

Following the standard procedure,[12] the spin configuration $\{\sigma, \mu\}$ on L and L_D can be mapped into vertex configurations on M. We do this by drawing bonds along the edges of M which separate spins of opposite signs. Then, as shown in Fig. 2, there are eight allowed configurations at a given vertex of M. A typical mapping of the configurations is shown in Fig. 3. The weights $w(\sigma\sigma', \mu\mu')$ are now considered as the vertex weights of M. Explicitly, we find

$$\begin{aligned} a &= w(+,+) = \omega_0 + \omega_1, \\ b &= w(-,-) = \omega_2 - \omega_3, \\ c &= w(-,+) = \omega_2 + \omega_3, \\ d &= w(+,-) = w_0 - \omega_1. \end{aligned} \tag{7}$$

Equation (5) now reads

$$Z_{\text{AT}}(\omega_0, \omega_1, \omega_2, \omega_3) = 2^{2-N_D} Z_{8v}(a, b, c, d), \tag{8}$$

where Z_{8v} is the generating partition function of the eight vertex model on M. The extra factor 2 in (8) is due to the 2 to 1 mapping of the spin and vertex configurations. The equivalence (8) is our main result and is valid for any *finite* planar lattice.

Several remarks are in order. First, we point out that the equivalence (8) is readily extended to the AT model with edge-dependent energies. In this case one arrives at a vertex model with site-dependent weights provided that (7) holds locally at all sites of M. Next we remark on the consequences of the equivalence on regular lattices. If L is a square lattice, then the medial lattice M is also a square lattice. However, the geometry of M is such that its vertices are rotated 90° from site to site.[13] It follows that the vertex weights c and d are interchanged from site to site and the resulting vertex model is "staggered".[6] Conversely, the Baxter eight vertex model[14] with the weights $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ uniformly at all sites is equivalent to an AT model whose interactions are different in the horizontal and vertical directions. Explicitly, we have

$$\begin{aligned} \omega_0^h &= \tfrac{1}{2}(\bar{a} + \bar{d}), & \omega_1^h &= \tfrac{1}{2}(\bar{a} - \bar{d}), \\ \omega_2^h &= \tfrac{1}{2}(\bar{b} + \bar{c}), & \omega_3^h &= \tfrac{1}{2}(\bar{b} - \bar{c}), \\ \omega_0^v &= \tfrac{1}{2}(\bar{a} + \bar{c}), & \omega_1^v &= \tfrac{1}{2}(\bar{a} - \bar{c}), \\ \omega_2^v &= \tfrac{1}{2}(\bar{b} + \bar{d}), & \omega_3^v &= \tfrac{1}{2}(\bar{b} - \bar{d}). \end{aligned} \tag{9}$$

If L is the triangular or honeycomb lattice, then M is the Kagomé lattice whose vertices are rotated 120° from site to site.[15] Now the 120° rotation is the only logical way of assigning directions and weights to the vertices of a Kagomé lattice. This suggests that the resulting eight vertex model on the Kagomé lattice is *regular* (as contrasted to *staggered*). Extending this reasoning to a square lattice, it is amusing to note that if the eight vertex model in (8) is considered *regular*, then it is the Baxter model that should be termed *staggered*!

Our result also leads to a duality relation for the AT model. Since going from L to L_D amounts to an interchange of $\{\sigma\}$ and $\{\mu\}$, or c and d in (7), we obtain the following duality relation for a *finite* AT model[9]:

$$2^{-N} Z_{\text{AT}}(\omega_0, \omega_1, \omega_2, \omega_3) = 2^{-N_D} Z_{\text{AT}}^{(D)}(\omega_0^*, \omega_1^*, \omega_2^*, \omega_3^*), \tag{10}$$

where $Z_{\text{AT}}^{(D)}$ is the partition function of the AT model on L_D with weights

$$\begin{aligned} \omega_0^* &= \tfrac{1}{2}(\omega_0 + \omega_1 + \omega_2 + \omega_3), & \omega_1^* &= \tfrac{1}{2}(\omega_0 + \omega_1 - \omega_2 - \omega_3), \\ \omega_2^* &= \tfrac{1}{2}(\omega_0 + \omega_2 - \omega_1 - \omega_3), & \omega_3^* &= \tfrac{1}{2}(\omega_0 + \omega_3 - \omega_1 - \omega_2). \end{aligned} \tag{11}$$

Extension of the duality (10) to the AT model with edge-dependent weights is obvious.

The vertex weight (6) can also be written as

$$w(\sigma\sigma', \mu\mu') = \exp(K_0^* + K_1^*\sigma\sigma' + K_2^*\mu\mu' + K_4^*\sigma\sigma'\mu\mu'), \tag{12}$$

with

$$\begin{aligned} \exp(4K_0^*) &= abcd, & \exp(4K_1^*) &= ab/cd, \\ \exp(4K_2^*) &= ac/bd, & \exp(4K_4^*) &= ad/bc. \end{aligned} \tag{13}$$

This is a generalization of the usual Ising representation[12] of the Baxter model, and the AT model is again pictured as two Ising models coupled with 4-spin interactions.

Finally, we remark on the boundary conditions. The equivalence (8) is valid for any finite planar lattice with a boundary. It also holds for planar lattices with cylindrical boundary conditions, for such lattices can be embedded in a plane and the dual can be defined accordingly. It is not clear how to extend the present result to planar lattices with toroidal boundary conditions. While the dual of such lattices can be defined in

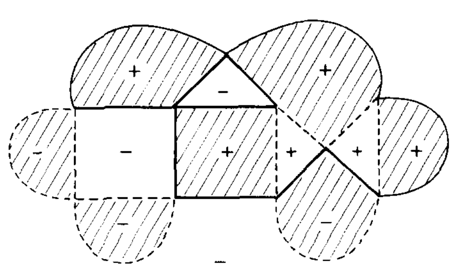

FIG. 3. The mapping of a typical spin configuration on L and L_D into a bond configuration on M.

an obvious way, the duality relation (4) for finite Ising models no longer holds.[16] Consequently, the equivalence (8) is not valid for such finite lattices.

For infinite lattices we expect the AT model to exhibit phase transitions. It has been pointed out[6] that from symmetry consideration alone two transitions are expected. Only when the two lower energies of ϵ_1, ϵ_2, and ϵ_3 become equal, do we expect the two transitions to coalesce into a single one. Furthermore, there are definitely two transitions in the parameter subspaces such as $K_3 = 0$ or $\epsilon_0 + \epsilon_3 = \epsilon_1 + \epsilon_2$. These are the basic ingredients in the consideration of Wu and Lin[17] in their construction of the critical surface for the AT model on the square lattice. We then expect the critical surface of the general AT model behave similarly in the parameter space. The critical surfaces would intersect at some single-transition trajectories lying in the sectional planes

$$\omega_i = \omega_j > \omega_k, \; i, \; j, \; k \text{ distinct}. \qquad (14)$$

The trajectories are straight lines for square lattice[17] and, very plausibly, parabolas for triangular and honeycomb lattices.[18,19] The three single-transition lines meet at the point $\omega_1 = \omega_2 = \omega_3 = \omega^*$, where ω^* is the critical point of the 4-component Potts model.

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