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LETTER TO THE EDITOR

Two phase transitions in triplet Ising models†

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Abstract. It is shown that the three-spin Ising model on a triangulated dice lattice is equivalent to an Ashkin–Teller model. We then conclude from the expected critical behaviour of the Ashkin–Teller model that the triplet model exhibits two phase transitions. Generalization of the consideration to other lattices and a suggestion on the possible order parameters associated with the transitions are given.

The exact solution of the three-spin (triplet) Ising model is now known as the Union Jack lattice (Hintermann and Merlini 1972), the triangular lattice (Baxter and Wu 1973, 1974, Baxter 1974), and a triangulated dice lattice (Wood 1973, Liu and Stanley 1974). A common property shared by these solutions is the occurrence of a unique transition at a temperature which can also be determined by a self-duality relationship. Recently, Wood and Pegg (1976) have extended the self-duality relation to a large class of triplet Ising models, thereby raising the question whether the critical point of *all* triplet models can be determined similarly. It is tempting to answer this question affirmatively, especially if the model is characterized by a single ferromagnetic interaction (Wood and Pegg 1976). The purpose of this Letter is to point out that such conclusion should be drawn with caution. We present evidence that in some triplet models there actually exist two phase transitions.

Consider the three-spin model on a triangulated dice lattice L introduced by Wood and Pegg (1976) as shown in figure 1. The lattice is constructed by completing *all* diagonals of the faces of a dice lattice, the latter being denoted by the heavy lines. Ising spins $\sigma = \pm 1$ are located at the sites of L and interact with three-spin interactions $-J\sigma\sigma'\sigma''$ around every elementary triangle $\triangle$. The partition function of this triplet model is

$$Z = \sum_{\{\sigma\}} \prod_{\Delta} \exp(K\sigma\sigma'\sigma''), \tag{1}$$

where $K = J/kT$.

Wood and Pegg (1976) showed that in the thermodynamic limit the free energy per site, f, of this (and any other) triplet model satisfies the duality relation

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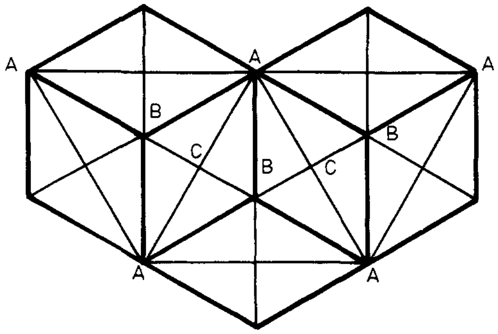

Figure 1. Triangulated dice lattice. The dice lattice is denoted by the heavy lines.

$$f(K) = \ln \sinh 2K + f(K^*), \tag{2}$$

where

$$\sinh 2K \sinh 2K^* = 1. \tag{3}$$

Assuming the existence of a unique transition, the Wannier (1945) argument then locates the critical point at $K = K^*$. We shall now show that there actually exist two transitions in this spin model. Consequently, the Wannier argument does not locate the transition point.

As shown in figure 1, the lattice L consists of three sublattices A, B and C. The sites of A form a triangular lattice, while the sites of B and C are located on a honeycomb lattice which is the dual of A. We first effect a dual transformation (Wegner 1973) for the spins on B and C. This introduces additional spins $\mu = \pm 1$ to the sites of A so that each site of A is now characterized by a four-valued variable (σ, μ). Furthermore, to each neighbouring pair (σ, μ) and (σ', μ') on A, the dual transformation yields a weight factor (Baxter and Wu 1973, 1974)

$$\omega(\sigma, \mu; \sigma', \mu') = 2^{-1/3} \{\exp [K(\sigma + \sigma')] + \mu\mu' \exp [-K(\sigma + \sigma')]\}^2. \tag{4}$$

The partition function (1) then reads

$$Z = \sum_{\{\sigma,\mu\}} \prod_{A} \omega(\sigma, \mu; \sigma', \mu'), \tag{5}$$

where the product is taken over all nearest neighbours of the triangular sublattice A.

It is now convenient to view the transformed spin model (5) as an Ashkin–Teller (1943) model. This is done by identifying the states $(+, +)$, $(+, -)$, $(-, +)$, $(-, -)$, respectively, as atoms A, B, C, D in an AT model (Fan 1972). Let the Boltzmann weights of the AT model be 1 between atoms of the same kinds, ω_1 between atoms AB or CD, ω_2 between atoms AC or BD, and ω_3 between atoms AD or BC. The partition function (5) is then precisely the partition function of an AT model on a triangular lattice with the following exact equivalence:

$$f(K) = \ln(2^{5/6} \cosh 2K) + \tfrac{1}{6} f_{\mathrm{AT}}(\omega_1, \omega_2, \omega_3). \tag{6}$$

Here f_{AT} is the free energy per site of the AT model with

$$\omega_1 = (\tanh 2K)^2 \qquad \omega_2 = (\cosh 2K)^{-2} \qquad \omega_3 = 0. \tag{7}$$

Now consider the analyticity of $f_{\mathrm{AT}}(\omega_1, \omega_2, \omega_3)$. Assuming a continuous dependence of the analytic behaviour of f_{AT} on ω_1, ω_2, ω_3 (the continuity assumption), the locus of the points at which f_{AT} fails to be analytic traces out a certain 'critical' surface Σ in the ω-space. As the temperature varies in a physical model, the parameters $(\omega_1, \omega_2, \omega_3)$ trace a thermodynamic path Γ in the ω-space. A phase transition is said to occur in the physical model whenever Γ intersects the surface Σ.

In an ordinary AT model where the ω's are themselves Boltzmann factors, Γ traces from (0, 0, 0) to (1, 1, 1) as the temperature rises in the AT model. In the present model, however, the relevant thermodynamic path is the straight line

$$\omega_1 + \omega_2 = 1 \qquad \omega_3 = 0 \tag{8}$$

connecting (1, 0, 0) and (0, 1, 0) shown in figure 2.

Two pieces of exact information are available for $\omega_3 = 0$. If ω_2 (or ω_1) also vanishes, then atoms A and B are decoupled from C and D in the AT model, and the system degenerates into an Ising model. Hence the surface Σ certainly passes through the two Ising critical points $(1/\sqrt{3}, 0, 0)$ and $(0, 1/\sqrt{3}, 0)$ denoted by B and C in figure 2. Also, the free energy $f_{\mathrm{AT}}(\omega_1, \omega_2, 0)$ can be evaluated exactly along the path $\omega_1^2 + \omega_2^2 = 1$, for it can be seen, using the arguments given above, that along this path the AT model is reducible to the three-spin Ising model on a triangular lattice. From the exact result of Baxter and Wu (1973, 1974), we conclude that Σ also passes through the point $(1/\sqrt{2}, 1/\sqrt{2}, 0)$. This exact critical point is denoted by A in figure 2.

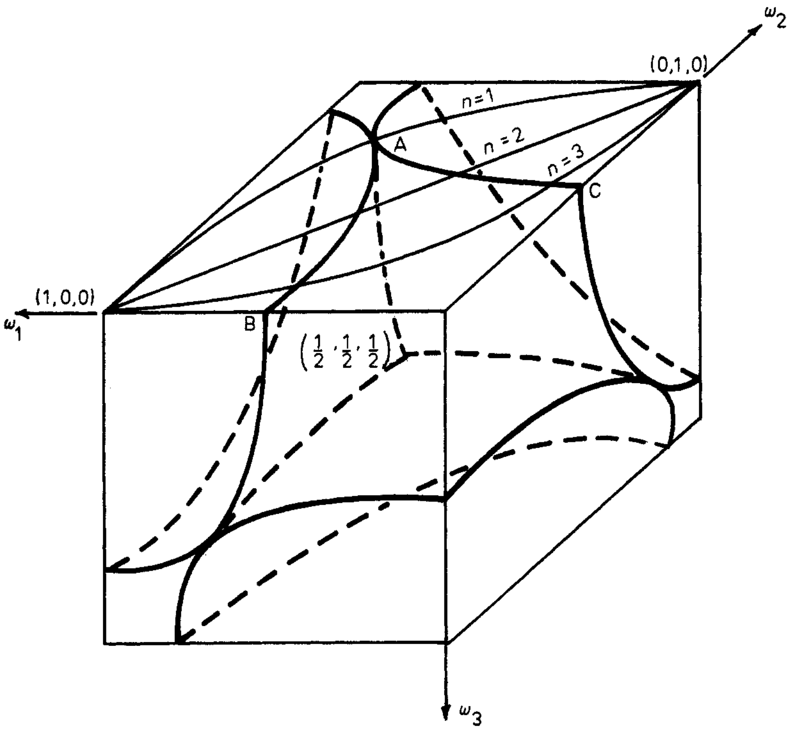

Figure 2. Schematic plot of the critical surface Σ for a triangular Ashkin–Teller model. The three bowl-shaped pieces join together at the parabolic segments (9). The thermodynamic paths for the triplet models trace from (1, 0, 0) to (0, 1, 0) along the path (10). The paths shown are for $n = 1, 2, 3$.

We are now able to give a schematic plot of the critical surface Σ for the triangular AT model. This is done in figure 2. The topology of Σ is similar to that of the square lattice (Wu and Lin 1974). Generally, any thermodynamic path leading from (0, 0, 0) to (1, 1, 1) will intersect Γ at two distinct point. The two transition points coalesce into a single point when $\omega_i = \omega_j \geqslant \omega_k$, and in this case the critical point lies on some single-transition trajectories (Wu 1976). For the square lattice, these single-transition trajectories are known to be straight lines (Wu and Lin 1974). In the present case of a triangular lattice, these trajectories originate from the Potts critical point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and terminate at the exact critical points $(1/\sqrt{2}, 1/\sqrt{2}, 0)$, $(1/\sqrt{2}, 0, 1/\sqrt{2})$, $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ established in the above. In fact, it has been conjectured (Enting 1975) on the basis of a self-dual relationship that these single-transition trajectories are the parabolas

$$2\omega_i^2 + \omega_k^2 = 1. \tag{9}$$

It is seen that these parabolas certainly pass through all the known critical points. We wish to emphasize that the validity of (9) is confined to the sectional planes $\omega_i = \omega_j \geqslant \omega_k$, i, j, k distinct. These parabolic segments are shown in figure 2 as the three broken lines intersecting at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Now the points A, B, C on $\omega_3 = 0$ are known to lie on Σ. Furthermore, it is expected that the point A is the only critical point on the line $\omega_1 = \omega_2$, $\omega_3 = 0$. It follows that the straight line (8) will always intersect Σ at two distinct points, provided that Σ is reasonably shaped. Consequently, we conclude that the triplet model (1) exhibits two phase transitions.

It might appear strange at first sight that a spin model characterized by a *single* ferromagnetic interaction should possess *two* transition points. However, such behaviour is not unreasonable for lattices consisting of nonequivalent sublattices. It is known, for example, that the two sublattice spontaneous magnetizations of the triplet model on the Union Jack lattice possesses different critical exponents (Wu 1975). In the present triplet model it can be seen using the argument of Watt and Enting (1975) that there is no high-temperature spontaneous magnetization. It is then reasonable to assume that the two transitions are accompanied by the onset of different sublattice spontaneous magnetizations. It would be useful to carry out a study of the series expansions for these sublattice magnetizations to test this suggestion. In fact, an analysis of the high (or low) temperature series of the free energy would serve the purpose of verifying the existence of two critical points.

Finally, we remark that the existence of two transitions is not unique to the triplet model in figure 1. For example, if the faces of L are further triangulated by adding more lines connecting the sites A in figure 1, we can show in a similar way that the triplet models on the resulting lattices are again reducible to a triangular AT model. Let each face of the dice lattice be divided into $2n$ faces; an example of $n = 3$ is shown in figure 3. Then the thermodynamic path of the triplet model traces from (1, 0, 0) to (0, 1, 0) along the path

$$\omega_1^{2/n} + \omega_2^{2/n} = 1 \qquad n = 1, 2, 3, \ldots. \tag{10}$$

We show the paths $n = 1, 2, 3$ in figure 2. It is seen that, except for $n = 1$ (the triangular lattice)' the triplet model always exhibits two phase transitions. The self-duality relation (2) merely reflects the intrinsic symmetry

$$f_{\mathrm{AT}}(\omega_1, \omega_2, 0) = f_{\mathrm{AT}}(\omega_2, \omega_1, 0) \tag{11}$$

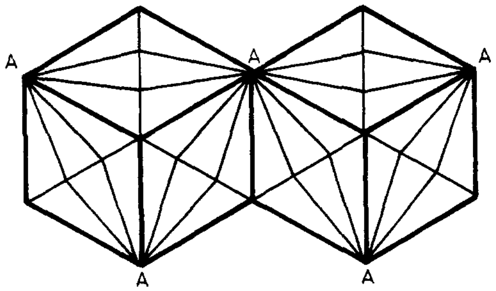

Figure 3. Example of a further triangulation ($n = 3$) of the lattice in figure 1.

in these models and, except for the case $n = 1$, does not locate the transition point. Similar triangulations of the Union Jack lattice can also be carried out and, using the same reasoning, the resulting triplet models (for even $n \geqslant 4$, however) will exhibit two phase transitions.

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