Duality transformation in a many-component spin model

F. Y. Wu* and Y. K. Wang

Department of Physics, Northeastern University, Boston, Massachusetts 02115
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It is shown that the duality transformation relates a spin model to its dual whose Boltzmann factors are the eigenvalues of the matrix formed by the Boltzmann factors of the original spin model. The duality relation valid for finite lattices is obtained, and applications are given.

The duality relation for two-dimensional spin models can be considered both from a topological and an algebraic point of view. A comprehensive discussion of these aspects for the Potts and the Ashkin-Teller (AT) models has been given by Mittag and Stephen.[1] More recently, Wegner[2] has reformulated the duality relation as an instance of a more general transformation. In this note we point out one further aspect of the duality transformations. Our result helps to clarify the reasoning in Wegner's formulation and also provides straightforward extensions of duality to other spin models.

Consider a q component spin model on a two-dimensional lattice L which has N sites. Let $\xi_i = 1, 2, \ldots, q$ denote the spin state of the ith site. The Hamiltonian can be generally written as

$$H = -\sum_{\langle ij \rangle} J(\xi_i, \xi_j), \tag{1}$$

where $-J(\xi, \xi')$ is the interaction between the spin states ξ and ξ'. The summation in (1) is over all interacting pairs $\langle ij\rangle$ which we assume to be noncrossing. The partition function is

$$Z = \sum_{\xi_i=1}^{q} \prod_{\langle ij \rangle} u(\xi_i, \xi_j), \tag{2}$$

with

$$u(\xi, \xi') = \exp[J(\xi, \xi')/kT]. \tag{3}$$

We shall restrict our attention to the case that

$$u(\xi_i, \xi_j) = u(\xi_i - \xi_j), \quad (\text{mod } q). \tag{4}$$

Thus the matrix U whose elements are $u(\xi, \xi')$ is cyclic. It is not necessary for our discussion to further assume that U is symmetric, although in most applications this will be the case. In order to distinguish ξ_i from ξ_j for a given edge connecting sites i and j, we place an arrow on it pointing from i to j. Thus the lattice is directed. We shall also have occasion to consider the situation, such as for the AT model, that U is block-cyclic. These cases will be explored in later discussions.

We can rewrite the partition function in two different ways. First, instead of specifying the spin states by ξ_i, we may label the edge in (4) by the difference $\xi_{ij} \equiv \xi_i - \xi_j$. However, to ensure that each set of ξ_{ij} will correspond to some spin states, it is necessary (and sufficient) to require

$$\sum_{\text{cw}} \xi_{ij} = \sum_{\text{ccw}} \xi_{ij} \tag{5}$$

around each face of L. Here the summation cw (ccw) is over the edges carrying clockwise (counterclockwise) arrows around the face. Clearly, the $\xi_i \Leftrightarrow \xi_{ij}$ mapping is q to 1. Denoting the restriction (5) by a prime over the summation sign, we can now rewrite the partition sum (2) as

$$Z(u) = q \sum_{\xi_{ij}=1}^{q}{}' \prod_{\langle ij \rangle} u(\xi_{ij}). \tag{6}$$

To make connection with the partition function on the dual of L, or L^D, we now cast Z into another form. Direct the edges of L^D such that the arrows on L^D coincide with those on L if each edge of L^D is rotated 90° clockwise. The situation around a site on L is shown in Fig. 1. Now the eigenvalues of the $q \times q$ cyclic matrix U are

$$\lambda(\eta) = \sum_{\xi=1}^{q} \exp(2\pi i \xi\eta/q)\, u(\xi), \quad \eta = 1, \ldots, q, \tag{7}$$

or, conversely,

$$u(\xi_{ij}) = \sum_{\eta=1}^{q} T(\xi_i, \eta)\, \lambda(\eta)\, T^*(\xi_j, \eta), \tag{8}$$

where

$$T(\xi, \eta) = q^{-1/2} \exp(2\pi i \xi\eta/q). \tag{9}$$

We substitute (8) into (2) and carry out the sums over ξ_i. At each site of L, we have for each outgoing (incoming) arrow a factor $T(\xi, \eta)$ $[T^*(\xi, \eta)]$. Denote the spin states of the spin model on L^D by η_α and identify the η in (8) as $\eta_{\alpha\beta} \equiv \eta_\alpha - \eta_\beta$, where the arrow runs from site α to site β on the corresponding edge of L^D. Then the summation over ξ_i (cf. Fig. 1) leads to a factor

$$\sum_{\xi_i=1}^{q} T(\xi_i, \eta_{21})\, T^*(\xi_i, \eta_{23}) \cdots T(\xi_i, \eta_{1n_i})$$
$$= q^{1-n_i/2}\, \delta_{\text{Kr}} \left(\sum_{\text{cw}} \eta_{\alpha\beta} - \sum_{\text{ccw}} \eta_{\alpha\beta} \right), \tag{10}$$

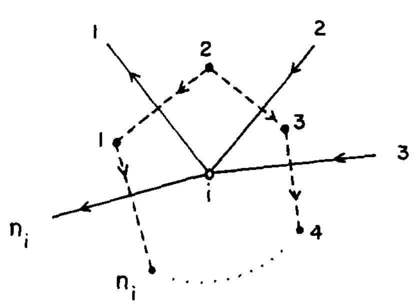

FIG. 1. The directed edges around the ith site on L. The solid (broken) lines are the edges of L (L^D).

where n_i is the number of neighbors of the ith site. The restriction imposed by the Kronecker delta on the rhs of (10) is exactly the same as in (5) for a face of L^D. Thus, after combining (8) with (2) and using (10), the partition function takes the form

$$Z = q^{N-E} \sum_{\eta_{\alpha\beta}=1}^{q}{}' \prod_{\langle\alpha\beta\rangle} \lambda(\eta_{\alpha\beta}), \tag{11}$$

where E is the number of edges of L (or L^D). Finally, by comparing (11) with (6) and using the Euler's relation for a connected planar graph,[3]

$$N + N_D = E + 2, \tag{12}$$

we obtain the identity

$$Z(u) = q^{1-N_D} Z^{(D)}(\lambda). \tag{13}$$

This is our main result and it is valid for any *finite* lattice. Here $Z^{(D)}(\lambda)$ is the partition function of the spin model on L^D whose Boltzmann factors are given by (7). While this result is implicit in Ref. 2, our discussion does bring out in a natural way the role played by the U matrix, thus clarifying the reasoning behind Wegner's formulation.

An example is the Potts model[4] with

$$U = \begin{pmatrix} e^K & 1 & \cdots & 1 \\ 1 & e^K & \cdots & 1 \\ & \cdots & \cdots & \\ 1 & 1 & \cdots & e^K \end{pmatrix}. \tag{14}$$

The eigenvalues of U are

$$\lambda_1 = e^K + q - 1, \quad \lambda_2 = \cdots = \lambda_q = e^K - 1, \tag{15}$$

so that the equivalence (13) reads

$$Z(e^K) = q^{1-N_D}(e^K - 1)^E Z^{(D)}(e^{K*}), \tag{16}$$

where

$$e^{K*} = \lambda_1/\lambda_2 = (e^K + q - 1)/(e^K - 1). \tag{17}$$

The above result is readily extended to the case where U is block-cyclic. An example is the AT model for which

$$U = \begin{pmatrix} U_1 & U_2 \\ U_2 & U_1 \end{pmatrix}, \tag{18}$$

where U_1 and U_2 are themselves 2×2 cyclic matrices.

Generally we consider a matrix U which is m-fold cyclic. That is to say, U is composed of q_1 cyclic matrices, each of which in turn contains q_2 cyclic matrices, etc., the dimension of U being $q = q_1 q_2 \cdots q_m$. Thus, an element of U, which specifies the spin states of the model, is described by an m component vector $\boldsymbol{\xi} \equiv (\xi_1, \ldots, \xi_m)$ whose components can take on, respectively, $q_1, q_2, \ldots, q_m$ different values. Treating the previous ξ and η as vectors, we can carry through all the steps and again arrive at the equivalence (13), provided that in place of (7) we have

$$\lambda(\boldsymbol{\eta}) = \sum_{\xi} \exp[2\pi i(\xi_1\eta_1/q_1 + \cdots + \xi_m\eta_m/q_m)]\, u(\boldsymbol{\xi}). \tag{19}$$

For the AT model we have $q_1 = q_2 = 2$, $\xi_i, \eta_i = 1, 2$. Equation (19) then leads to the duality relations derived by Ashkin and Teller.[5] As a further illustration consider the six-component spin model whose U matrix is

$$U = \begin{pmatrix} U_1 & U_2 & U_2 \\ U_2 & U_1 & U_2 \\ U_2 & U_2 & U_1 \end{pmatrix}, \tag{20}$$

where $U_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and $U_2 = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ are 2×2 matrices. It is easily seen that the eigenvalues of U form a similar cyclic matrix whose elements are

$$\begin{aligned} a^* &= \lambda_1 = a + b + 2(\alpha + \beta), \\ b^* &= \lambda_2 = a - b + 2(\alpha - \beta), \\ \alpha^* &= \lambda_3 = \lambda_4 = a + b - (\alpha + \beta), \\ \beta^* &= \lambda_5 = \lambda_6 = a - b - (\alpha - \beta). \end{aligned} \tag{21}$$

This is the duality transformation.

Note added in proof: Finally we remark that our result (13) is valid even if the Boltzmann factor (3) is edge-dependent. In this case the eigenvalues (7) or (19) are introduced for each edge ij and in (13) we have $u = \{u_{ij}\}$, $\lambda = \{\lambda_{ij}\}$.

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