Eight-vertex model on the honeycomb lattice*

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The most general vertex model defined on a honeycomb lattice is the eight-vertex model. In this paper it is shown that the symmetric eight-vertex model reduces to an Ising model with a nonzero real or pure imaginary magnetic field H. The equivalent Ising model is either ferromagnetic with $e^{2H/kT}$ real or antiferromagnetic with $e^{2H/kT}$ unimodular. The exact transition temperature and the order of phase transition in the former case are determined. As an application of the result we verify the absence of a phase transition in the monomer-dimer system on the honeycomb lattice.

1. INTRODUCTION

The vertex model in statistical mechanics plays an important role in the study of phase transitions in lattice systems. A case of current interest is the eight-vertex model on a square lattice.[1,2] This is a rather special model in which only a limited number of the possible vertex types are allowed. The most general one on a square lattice would be the sixteen-vertex model.[3] Unfortunately, except in some special cases,[4,5] the behavior of this general model is not known.

In this paper we consider the counterpart of the sixteen-vertex model of a square lattice for the honeycomb lattice. That is, we consider an eight-vertex model defined on the hexagonal lattice. It turns out that we can say a lot more in this case. While the exact solution of this model still proves to be elusive in most cases, we can make definite statements about its phase transition. In particular, the exact transition temperature can be quite generally determined. An application of our result is the verification of the absence of a phase transition in the monomer-dimer system on the honeycomb lattice.

2. DEFINITION OF THE MODEL

In the study of a vertex model one is interested in the evaluation of a graph generating function. Consider a honeycomb lattice and draw bonds (graphs) along the lattice edges such that each edge can be independently "traced" or left "open." Denote the traced (resp. open) edges by solid (resp. broken) lines; then, as shown in Fig. 1, there are eight possible vertex configurations. With each type of vertex configuration we associate a vertex weight a, b, c, or d (see Fig. 1). Our object is to evaluate the generating partition function

$$Z = Z(a, b, c, d) = \sum_G a^{n_0} b^{n_1} c^{n_2} d^{n_3}, \qquad (1)$$

where the summation is over all possible graphs on the lattice and, for a given graph G, n_i is the number of vertices having i solid lines (or bonds). This defines an "eight-vertex" model for the honeycomb lattice.

Since all possible vertex types are allowed, this eight-vertex model is the counterpart of the sixteen-vertex model of a square lattice. Note that we do not distinguish the bonds in different directions. Whereas it is possible to consider the further generalization of eight different weights, we shall not go into this complication in this paper. As a motivation we point out some special cases of interest. When $c=d=0$, the partition function (1) becomes the monomer-dimer generating function for the honeycomb lattice. When $b=d=0$, Z reduces to the partition function of a zero-field Ising model, which can be evaluated by pfaffians.

In a statistical model of phase transitions, the vertex weights are the Boltzmann factors

$$a=\exp(-\epsilon_0/kT), \quad b=\exp(-\epsilon_1/kT),$$
$$c=\exp(-\epsilon_2/kT), \quad d=\exp(-\epsilon_3/kT) \qquad (2)$$

where ϵ_i is the energy of a vertex having i bonds. While the weights (2) are always positive, the symmetry relations to be derived below are valid more generally for any real or complex weights.

3. SYMMETRY RELATIONS

The partition function (1) possesses a number of symmetry properties. Interchanging the solid and broken lines in Fig. 1, we obtain the symmetry relation

$$Z(a, b, c, d) = Z(d, c, b, a). \qquad (3)$$

Also since both the total number of vertices, N, and the number of vertices with odd number of bonds are even, we have the negation symmetry

$$Z(a, b, c, d) = Z(-a, -b, -c, -d)$$
$$= Z(-a, b, -c, d)$$
$$= Z(a, -b, c, -d). \qquad (4)$$

The weak graph expansion[6] yields an additional symmetry relation. For its derivation it is most convenient to use Wegner's formulation[7] of the weak-graph expansion. Denote the vertex weights by $\omega(i,j,k)$, where $i,j,k=\pm 1$ are the edge indices such that $+1$ corresponds to no bond and -1 corresponds to a bond on the edge. I.e., $\omega(+,+,+)=a$, $\omega(+,+,-)=\omega(+,-,+)=\omega(-,+,+)=b$, $\omega(+,-,-)=\omega(-,+,-)=\omega(-,-,+)=c$, and $\omega(-,-,-)=d$. Define a set of new vertex weights $\omega^*(+,+,+)=a^*$, etc. by

$$\omega^*(\alpha,\beta,\gamma) = \sum_{ijk} V_{\alpha i} V_{\beta j} V_{\gamma k}\, \omega(i,j,k), \qquad (5)$$

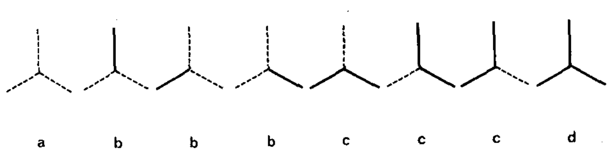

FIG. 1. The eight vertex configurations and the associated weights for a honeycomb lattice.

where the 2×2 matrix V having elements $V_{\alpha i}$ satisfies

$$V\tilde{V}=I, \tag{6}$$

I being the identity matrix. We then have the weak-graph symmetry

$$Z(a,b,c,d)=Z(a^*,b^*,c^*,d^*). \tag{7}$$

There are two possible choices for V:

$$V(y)=(1+y^2)^{-1/2}\begin{pmatrix}1 & y\\ y & -1\end{pmatrix} \tag{8}$$

or

$$U(y)=(1+y^2)^{-1/2}\begin{pmatrix}1 & y\\ -y & 1\end{pmatrix} \tag{9}$$

for aribitrary (real or complex) y. The explicit transformation generated by (8) is

$$\begin{aligned}
a^* &= (1+y^2)^{-3/2}[a+3yb+3y^2c+y^3d],\\
b^* &= (1+y^2)^{-3/2}[ya-(1-2y^2)b+(y^3-2y)c-y^2d],\\
c^* &= (1+y^2)^{-3/2}[y^2a+(y^3-2y)b+(1-2y^2)c+yd],\\
d^* &= (1+y^2)^{-3/2}[y^3a-3y^2b+3yc-d].
\end{aligned} \tag{10}$$

The transformation generated by (9) leads to identical vertex weights subject to the negation symmetry $b^*\to -b^*$; $d^*\to -d^*$ hence is not independent. We shall write (10) in the short-hand notation

$$\omega^*(y)=V(y)\omega. \tag{11}$$

It is also seen that two consecutive transformations are equivalent to a single one:

$$V(x)V(y)=U\left(\frac{y-x}{1+xy}\right). \tag{12}$$

In particular we have

$$V(y)V(y)=I. \tag{13}$$

4. SPECIAL SOLUTIONS

Before we consider the model with general weights, it is useful to first consider some special cases whose solutions are known

A. $b=ua,\ c=u^2a,\ d=u^3a$

The vertex weights in this case can be converted into the bond weight u^2. Since *all* graphs are included in (1), we then obtain

$$\begin{aligned}
Z&=a^NZ(1,u,u^2,u^3)\\
&=a^N(1+u^2)^{3N/2}.
\end{aligned} \tag{14}$$

Here we see a simple example for which the partition function (1) does not exhibit a phase transition.

B. $b=d=0$

Here only the vertices with even number (0 or 2) of bonds are allowed. The graphs in (1) are then precisely those encountered in the high-temperature expansion of a zero-field Ising model. Writing

$$c/a=\tanh K, \tag{15}$$

we then obtain

$$\begin{aligned}
Z&=Z(a,0,c,0)\\
&=a^NZ(1,0,\tanh K,0)\\
&=a^N2^{-N}(\cosh K)^{-3N/2}Z_{\text{Ising}}(0,K),
\end{aligned} \tag{16}$$

where more generally $Z_{\text{Ising}}(L,K)$ is the partition function of an Ising model on the honeycomb lattice with interactions $-kTK$ and a magnetic field $-kTL$. From the known expression[8] of $Z_{\text{Ising}}(0,K)$ given by (A1) we obtain, in the large N limit,

$$\begin{aligned}
\frac{1}{N}\ln Z=(16\pi^2)^{-1}\int_0^{2\pi}d\theta\int_0^{2\pi}d\phi\ln\{a^4+3c^4+2(c^4-a^2c^2)\\
\times[\cos\theta+\cos\phi+\cos(\theta+\phi)]\}.
\end{aligned} \tag{17}$$

We remark that (17) is valid for arbitrary (real or complex) a and c, although the physical range of an Ising model is restricted to real values satisfying $|c/a|\leq 1$. The expression (17) is nonanalytic at

$$a/c=\pm\sqrt{3}. \tag{18}$$

Other established properties of $Z_{\text{Ising}}(L,K)$ for $L\neq 0$ are summarized in the Appendix.

C. $a=d,\ b=c$

The vertex weights are now symmetric under the interchange of the solid and the broken lines in Fig. 1. In this case we can again reduce the partition function to the form of (16). Indeed, taking $y=1$ in (10), we obtain

$$Z=Z((a+3b)/\sqrt{2},0,(a-b)/\sqrt{2},0). \tag{19}$$

The phase transition now occurs at

$$a/b=3\pm 2\sqrt{3}. \tag{20}$$

D. $ad=bc$

In this case we define the Ising parameters L and K by

$$z=\tanh K=c/a,\quad \tau=\tanh L=b/\sqrt{ac}. \tag{21}$$

Then

$$\begin{aligned}
Z&=a^NZ(1,\sqrt{z}\tau,z,z^{3/2}\tau)\\
&=a^N2^{-N}(\cosh L)^{-N}(\cosh K)^{-3N/2}Z_{\text{Ising}}(L,K)\\
&=(2a^3c)^{-N}(ac-b^2)^N(a^2-c^2)^{3N/2}Z_{\text{Ising}}(L,K).
\end{aligned} \tag{22}$$

Here the second step follows from the generalization of (16) to the high-temperature expansion of $Z_{\text{Ising}}(L,K)$.

E. $b^2=ac$

In this case we have

$$Z=a^NZ(1,u^{-1},u^{-2},d/a), \tag{23}$$

where $u=a/b$. The partition function on the rhs of (23) is in a form similar to that considered in Ref. 5. We then obtain in a similar fashion[9]

$$Z=(b/a)^{2N}(1+a^2/b^2)^{3N/8}(ad/bc-1)^{N/2}Z_{\text{Ising}}(L,K), \tag{24}$$

where

$$\begin{aligned}
\exp(4K)&=1+a^2/b^2,\\
\exp(2L)&=(1+a^2/b^2)^{3/2}(ad/bc-1)^{-1}.
\end{aligned} \tag{25}$$

We see that the Ising model is ferromagnetic for real

a/b. For the Boltzmann weights (2) (subject to $2\epsilon_1 = \epsilon_0 + \epsilon_2$), we find the model in general exhibits no phase transition, except for $\epsilon_0 < \epsilon_1 (a > b)$ and $\gamma_0 < (\epsilon_3 - \epsilon_0)(\epsilon_1 - \epsilon_0)^{-1} < 0$ the model has a first-order phase transition at $\exp(2L) = 1$ or

$$(a^2 + b^2)^{3/2} = a^2 d - b^3. \tag{26}$$

Here $\gamma_0 = 3 - 2\ln(27 + 15\sqrt{3})/\ln(6 + 4\sqrt{3}) = -0.102\,2204\cdots$.

5. GENERAL CASE

We are now in a position to discuss the general solution for arbitrary (positive) vertex weights (2). The idea is to introduce the weak-graph transformation (10) and choose y to make the new vertex weights satisfying either $a^*d^* = b^*c^*$ or $b^{*2} = a^*c^*$. We can then use the results of the Appendix to determine the critical behavior of the vertex model. For clarity we use subscripts 1 and 2 to distinguish the two cases. That is, in analogy to (11), we write

$$\omega_i^* \equiv \omega^*(y_i) = V(y_i)\omega, \quad i = 1, 2, \tag{27}$$

and consider the two cases separately.

(i) $a_1^* d_1^* = b_1^* c_1^*$: From (27) and (10) we find y_1 given by

$$y_1^2 - 2Ay_1 - 1 = 0, \tag{28}$$

where $A = (b^2 - ac + bd - c^2)/(ad - bc)$. The new vertex weights $\omega_1^* = \{a_1^*, b_1^*, c_1^*, d_1^*\}$ are real if we take the positive solution

$$y_1 = A + (A^2 + 1)^{1/2} > 0. \tag{29}$$

Then, from (10), $a_1^* > 0$. Also c_1^* is real since

$$a_1^* + c_1^* = (1 + y_1^2)^{-1/2}(a + by_1 + c + dy_1) > 0. \tag{30}$$

The partition function is now

$$Z = (2a_1^{*3}c_1^*)^{-N}(a_1^*c_1^* - b_1^{*2})^N (a_1^{*2} - c_1^{*2})^{3N/2} \times Z_{\text{Ising}}(L_1^*, K_1^*), \tag{31}$$

where

$$\exp(2K_1^*) = (a_1^* + c_1^*)/(a_1^* - c_1^*),$$
$$\exp(2L_1^*) = [(a_1^*c_1^*)^{1/2} + b_1^*]/[(a_1^*c_1^*)^{1/2} - b_1^*]. \tag{32}$$

We observe that $\exp(2K_1^*) < 1$, $L_1^* =$ pure imaginary if $c_1^* < 0$. Since not much is known about $Z_{\text{Ising}}(L_1^*, K_1^*)$ for K_1^* and L_1^* in these ranges, we shall be interested only in $c_1^* > 0$. We observe in particular that, for a_1^* and c_1^* positive, $\exp(2L_1^*) \neq -1$.

(ii) $b_2^{*2} = a_2^*c_2^*$: From (27) and (10) we find y_2 given by

$$(bd - c^2)y_2^2 + (ad - bc)y_2 + (ac - b^2) = 0. \tag{33}$$

The partition function is then

$$Z = (b_2^*/a_2^*)^{2N}(1 + a_2^{*2}/b_2^{*2})^{3N/8} \times (a_2^*d_2^*/b_2^*c_2^* - 1)^{N/2} Z_{\text{Ising}}(L_2^*, K_2^*). \tag{34}$$

Here the weights $a_2^*, b_2^*, c_2^*, d_2^*$ are real if the discriminant

$$\Delta \equiv (ad - bc)^2 - 4(bd - c^2)(ac - b^2) \tag{35}$$

is positive. The parameters K_2^* and L_2^* are given by (25) with $a \to a_2^*$, etc. After some steps we find the simple result:

$$\exp(4K_2^*) = 1 + \Delta/(bd - c^2 + ac - b^2)^2 > 0. \tag{36}$$

We shall consider $\Delta > 0$ which corresponds to K_2^* being ferromagnetic. The similar expression of L_2^*, which is not needed for our discussions, is rather complicated and will not be given.

The two transformations (i) and (ii) are obviously related. To see the relationship, we observe from (27), (12), and (13) that

$$\omega_2^* = V(y_2)V(y_1)\omega_1^* = U\left(\frac{y_1 - y_2}{1 + y_1y_2}\right)\omega_1^*. \tag{37}$$

Since (34) is invariant under the negation of b_2^* and d_2^*, there exists a single transformation which relates ω_1^* to ω_2^*. To effect this transformation, we set $ad = bc$ in (33) and obtain $y_2 = (a/c)^{1/2}$. The new weights are then

$$\begin{aligned} a_2^* &= 4(1 + a/c)^{-3/2}(a/c)^{1/2}(b + \sqrt{ac}), \\ b_2^* &= 2(1 + a/c)^{-3/2}(a/c - 1)(b + \sqrt{ac}), \\ c_2^* &= b_2^{*2}/a_2^*, \\ d_2^* &= (1 + a/c)^{-3/2}(a^{5/2}/c^{3/2} - 3ab/c + 3\sqrt{ac} - bc/a). \end{aligned} \tag{38}$$

Now (36) becomes, for $ad = bc$,

$$\exp(4K_2^*) = [(a + c)/(a - c)]^2. \tag{39a}$$

Also using (38), we find

$$\begin{aligned} \exp(2L_2^*) &= (\sqrt{ac} + b)/(\sqrt{ac} - b), \quad \text{if } a/c > 1, \\ &= (b + \sqrt{ac})/(b - \sqrt{ac}), \quad \text{if } a/c < 1. \end{aligned} \tag{39b}$$

Letting $a = a_1^*$, $b = b_1^*$, $c = c_1^*$, $d = d_1^*$ in (39) and comparing with (32), we then obtain the relation

$$\begin{aligned} \exp(4K_2^*) &= \exp(4K_1^*), \\ \exp(2L_2^*) &= \pm\exp(2L_1^*), \quad \text{for } a_1^*/c_1^* \gtrless 1. \end{aligned} \tag{40}$$

Note that while $\exp(2K_2^*)$ can be taken to be positive, $\exp(2K_1^*)$ can be either positive or negative. We observe from (40), (32), and (36) that $\Delta > 0$ and $c_1^* > 0$ are equivalent. Hence, for $\Delta > 0$, K_2^* is ferromagnetic and $\exp(2L_2^*)$ is real.

Using the results of the Appendix, we conclude that, for $\Delta > 0$, the nonanalyticity of Z can occur only at $\exp(2L_2^*) = +1$ or -1. To distinguish these two cases, we turn to L_1^*. Since $\exp(2K_1^*)$ may be negative, it is then convenient to consider the following situations separately:

(i) $a_1^* > c_1^* > 0$: From (40) and $\exp(2L_1^*) \neq -1$, the nonanalyticity can occur only at $\exp(2L_1^*) = \exp(2L_2^*) = 1$. By using (32) this is equivalent to

$$b_1^* = d_1^* = 0. \tag{41}$$

A little algebra using (28) reduces (41) to

$$2(ab - cd)[(b^2 - ac + bd - c^2)^2 - (ad - bc)^2] + (ad - bc)(b^2 - ac + bd - c^2) \times (a^2 + d^2 - 3b^2 - 3c^2 - 2ac - 2bd) = 0 \tag{42}$$

which defines $T = T_c$. To see whether indeed a phase transition occurs at T_c, we observe that K_1^* and K_2^* are equal and positive. Then from the result of the Appendix we need to compute $z_c = (c_1^*/a_1^*)_{T=T_c}$. The vertex model will exhibit a first-order transition if $z_c > 1/\sqrt{3}$, a second-order transition with an infinite specific heat if $z_c = 1/\sqrt{3}$, and no transition at all if $z_c < 1/\sqrt{3}$, even if (42) has a solution. The following useful expression of z_c is obtained by combining (29), (10) and (41):

$$z_c = \left[\frac{4(ac+bd)A^2 + 4(ab-cd)A + (3b+d)(d-b) + (a+3c)(a-c)}{4(a^2+d^2)A^2 + 12(ab-cd)A + (3b+d)^2 + (a+3c)^2}\right]_{T_c}. \tag{43}$$

(ii) $c_1^* > a_1^* > 0$: In this case the nonanalyticity occurs only at $\exp(2L_2^*) = -\exp(2L_1^*) = -1$. Then T_c is again given by (41) or (42). Now $K_2^* > 0$ and $\exp(2L_2^*) = -1$; hence the vertex model always has a first-order transition. Note that we can reach the same conclusion by considering K_1^*. In this case $\exp(2K_1^*) < -1$ and $\exp(2L_1^*) = 1$. We need only to reverse the signs of $\exp(2K_1^*)$ and $\exp(2L_1^*)$ which leaves $Z_{\text{Ising}}(L_1^*, K_1^*)$ unchanged, as can be seen from the low-temperature expansion.

Combining the results in (i) and (ii), we conclude that a phase transition occurs for $\Delta > 0$ only if $z_c \geq 1/\sqrt{3}$.

A special case is that (41) or (42) is an identity. Then, for *all* Δ, $L_1^* = 0$ and Z reduces to that of a zero-field Ising model. The vertex model now exhibits the Ising-type transition (logarithmic specific heat singularity) at T_c defined by

$$\Delta/(bd - c^2 + ac - b^2)^2 = (2+\sqrt{3})^{\pm 2} - 1. \tag{44}$$

Unfortunately we are unable to make any general statement for $\Delta < 0$. For $\Delta < 0$, K_2^* is antiferromagnetic and $\exp(2L_2^*)$ is unimodular and lies on the unit circle. Presumably the zeros of an Ising antiferromagnet also distribute along the unit circle in the thermodynamic limit.[10] The vertex model then in general shows a unique transition.

6. SUMMARY

We have established the following results for the vertex model (2):

(i) If (42) is an identity, then an Ising-type transition occurs at T_c defined by (44), where Δ is given in (35).

(ii) For $\Delta \geq 0$ and (42) not an identity, a phase transition occurs at T_c defined by (42) if $z_c \geq 1/\sqrt{3}$, where z_c is given in (43). Otherwise ($z_c < 1/\sqrt{3}$) there is no phase transition. The transition is of first-order except that he specific heat diverges for $z_c = 1/\sqrt{3}$.

(iii) For $\Delta < 0$ and (42) not an identity, the vertex model is related to an Ising antiferromagnet with a pure imaginary magnetic field. Nature of the transition is not known.

It is instructive to illustrate with some examples.

(i) $a = d$, $b = c$: Since (42) is an identity, we find from (44) the critical condition

$$(a^2 + 2ab - 3b^2)/4b^2 = (2 \pm \sqrt{3})^2 - 1,$$

which agrees with (20).

(ii) $b^2 = ac$: We find $\Delta = (a^2 d - b^3)^2/a^2 > 0$ and $\exp(4K_2^*) = 1 + a^2/b^2$. This is in agreement with (25). It can be verified that the condition (42) is the same as that obtained from $\exp(2L) = 1$ in (25).

(iii) $b = c = d$: We find $\Delta = b^2(a-b)^2 > 0$ and $\exp(4K_2^*) = 2$. Since K_2^* is a constant with $z_c^{-1} = 3 + 2\sqrt{2} > \sqrt{3}$, there is no phase transition.

(iv) *Monomer-dimer system*: For $c = d = 0$ the partition function (1) becomes the monomer-dimer generating function $Z_{\text{MD}}(a, b^2)$ where a and b^2 are, respectively, the monomer and dimer activities. It is known that this system does not have a phase transition.[11] We verify this by observing that $\Delta = 0$, $K_2^* = 0$. Also (42) has no solution for $c = d = 0$, $ab \neq 0$.

To obtain a closed expression for Z_{MD}, we find that, for $c = d = 0$, either $\exp(2K_2^*) = 1$, $\exp(2L_2^*) = -1$ or $\exp(2K_1^*) = -1$, $\exp(2L_1^*) = 1$. In either case the Ising partition function is identically zero. Therefore we must take the limit $c = d \to 0$ appropriately. This leads to the expression

$$Z_{\text{MD}}(a, b^2) = \lim_{c \to 0} (b/4c)^N Z_{\text{Ising}}(L_2^*, K_2^*) \tag{45}$$

where (for small c)

$$\exp(2K_2^*) = 1 + 4c/b,$$

$$\exp(2L_2^*) = -1 \pm 2a\sqrt{c}/b^{3/2}.$$

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APPENDIX: ISING PARTITION FUNCTION

We summarize in this Appendix the relevant properties of the Ising partition function $Z_{\text{Ising}}(L, K)$.

A closed expression is known for $L = 0$. In the large N limit, one has[8]

$$\frac{1}{N}\ln Z_{\text{Ising}}(0, K) = \tfrac{3}{4}\ln 2 + \frac{1}{16\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \times \ln[c^3 + 1 - s^2(\cos\theta + \cos\phi + \cos(\theta+\phi))], \tag{A1}$$

where

$$c = \cosh 2K, \quad s = \sinh 2K.$$

The second derivative of (A1) diverges logarithmically at $\tanh K = \pm 1/\sqrt{3}$.

A unique property of the honeycomb lattice (coordination number = odd) is that the partition functions at $L = i\frac{1}{2}\pi$ and $L = 0$ are related. To see this connection, consider the high-temperature expansion of $Z_{\text{Ising}}(L, K)$. Using the identities for $L = i\frac{1}{2}\pi$,

$$\sum_{\sigma=\pm 1} \sigma \exp(L\sigma) = 2\sinh L = 2i, \tag{A2}$$

$$\sum_{\sigma=\pm 1} \exp(L\sigma) = 2\cosh L = 0,$$

we see that only the vertices with odd number of bonds contribute in the expansion. Thus we obtain

$$Z_{\text{Ising}}(i\tfrac{1}{2}\pi, K) = (2i)^N (\cosh K)^{3N/2} Z(0, \sqrt{z}, 0, z^{3/2}) = Z_{\text{Ising}}(0, \widetilde{K}), \quad \text{(A3)}$$

where

$$\tanh\widetilde{K} \tanh K = 1.$$

The last step follows from the symmetry relation (3) and (16). Note that $Z_{\text{Ising}}(i\frac{1}{2}\pi, K)$ is analytic for real K.

Most of the established properties for $L \neq 0$ are for ferromagnetic interactions ($K > 0$). For $K > 0$, $Z_{\text{Ising}}(L, K)$ can be nonanalytic in L or K only at $|\exp(2L)| = 1$.[12,13] This means $\exp(2L) = \pm 1$ for real $\exp(2L)$. At $\exp(2L) = 1$ the analyticity extends to all $0 < z < 1/\sqrt{3}$ while the first derivative w.r.t. L is discontinous for all $1/\sqrt{3} < z < 1$. At $\exp(2L) = -1$ this first derivative is presumably discontinuous for all $0 < z < 1$. This is similar to the result of a square lattice[14] and can be easily seen to hold in both the high and low temperature limits. We hope to return in the future for an exact calculation of this discontinuity.

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