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LETTER TO THE EDITOR

Two phase transitions in the Ashkin–Teller model†

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Abstract. Assuming a continuous dependence of the criticality on the energy parameters, it is conjectured that the Ashkin–Teller model has in general two phase transitions. The two critical points coalesce into a single one when the two middle energies in the model are degenerate.

Some 30 years ago Ashkin and Teller (1943) introduced a lattice statistical model, now known as the Ashkin–Teller (AT) model, as a generalization of the Ising model to a four-component system. Using the Kramers–Wannier (1941) argument and the assumption of a unique transition, they conjectured the location of the critical point for a special case of this model in which three of the four components are degenerate. Their conjecture has recently been extended to the general AT model (Fan 1972a), but Wegner (1972) showed that the argument does not in general locate the critical point. It is therefore worthwhile to examine more closely the problem of the existence of phase transitions in the AT model. Here, on the basis of established results and a plausible continuity assumption on the criticality, we conjecture that the AT model has in general two phase transitions. Only in a special case do the two transitions coalesce into a single one.

The AT model is defined on a square lattice whose sites are occupied by any of the four kinds of atoms A, B, C and D. Two neighbouring atoms interact with an energy: ϵ_0 for AA, BB, CC, DD; ϵ_1 for AB, CD; ϵ_2 for AC, BD; and ϵ_3 for AD, BC. By relabelling the atoms, on a sublattice basis if necessary, one sees that the four ϵ's can be freely permuted. This is a basic symmetry of the AT model.

A useful spin representation (Fan 1972b) of the AT model is as follows. At each site of the square lattice one introduces two Ising spins so as to form a two-layer lattice. The four nearest-neighbouring spins interact with a four-body interaction $-J_3 = \frac{1}{4}(\epsilon_0 + \epsilon_3 - \epsilon_1 - \epsilon_2)$ and two-body interactions $-J_1 = \frac{1}{4}(\epsilon_0 + \epsilon_1 - \epsilon_2 - \epsilon_3)$ and $-J_2 = \frac{1}{4}(\epsilon_0 + \epsilon_2 - \epsilon_1 - \epsilon_3)$ within each layer. The basic symmetry of the ϵ's then implies that J_1, J_2 and J_3 can also be permuted.

By performing a dual transformation for the Ising spins in one layer and interpreting the result as a vertex model (Wu 1971, Kadanoff and Wegner 1971), Wegner (1972)

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showed that the AT model is equivalent to a *staggered* eight-vertex model. To be specific, the equivalent vertex model has vertex weights a, b, c_+, d_+ on sublattice A and a, b, d_+, c_+ on sublattice B, with

$$\begin{aligned} a &= 2^{-1/2}(\omega_0 + \omega_1), & b &= 2^{-1/2}(\omega_2 - \omega_3), \\ c_+ &= 2^{-1/2}(\omega_2 + \omega_3), & d_+ &= 2^{-1/2}(\omega_0 - \omega_1). \end{aligned} \tag{1}$$

Here $\omega_i = \exp(-\epsilon_i/kT)$ and we have adopted Baxter's notations (Baxter 1971) a, b, c, d for the vertex weights.

From the basic symmetry we may take, without loss of generality, $\omega_0 = 1$ and ω_1, ω_2, ω_3, $\leqslant 1$. As the temperature varies from 0 to ∞, the point (ω_1, ω_2, ω_3) traces in the ω-space a curve, the thermodynamic path Γ, from (0, 0, 0) to (1, 1, 1). Assuming a continuous dependence of the criticality on the parameters ω_1, ω_2 and ω_3 (the continuity assumption), the critical point of the AT model will trace a surface σ in the ω-space. A phase transition occurs whenever Γ intersects the surface σ. This continuity assumption appears to have been first clearly stated by Thibaudier and Villain (1972) in connection with locating the critical point for the three-dimensional eight-vertex model.

Some exact information is available for the critical surface σ. When any one of the Ising interactions J_1, J_2 and J_3 vanishes, the Ising representation of the AT model decouples into two independent nearest-neighbour models. From the exact result on the Ising model (Onsager 1944) we conclude that the points

$$J_i = 0, \qquad \exp(-2|J_j|/kT) = r \equiv \sqrt{2} - 1, \qquad i \neq j \tag{2}$$

lie on σ. A little algebra shows that in the ω-space these are the straight lines L, shown in figure 1, connecting the points (r, 0, 0), (r, 1, r), (0, 0, r), (1, r, r), (0, r, 0), (r, r, 1), (r, 0, 0) in succession.

From the eight-vertex representation (1) of the AT model, it is known from Baxter's work (Baxter 1971) that when $c_+ = d_+$ the critical condition is $a = |b| + c_+ + d_+$.

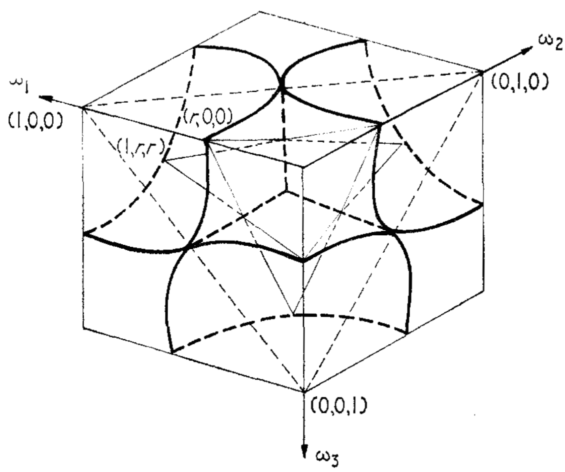

Figure 1. Schematic plot of the critical surface σ in the ω-space. The light lines L lie on σ, and the heavy broken lines L_1 are the intersections of σ with the plane $\omega_1 + \omega_2 + \omega_3 = 1$. The cube $0 \leqslant \omega_i \leqslant 1$ indicates the physical region.

Coupled with the basic symmetry of the AT model we then conclude that σ intersects the plane

$$\omega_1 + \omega_2 + \omega_3 = 1 \tag{3}$$

at the line segments L_1

$$\omega_i = \omega_j \geqslant \omega_k, \qquad i, j, k \text{ distinct.} \tag{4}$$

These segments are also shown in figure 1. Note that L_1 and two of the lines L meet at the common points (r^2, r, r), (r, r^2, r) and (r, r, r^2).

When $\epsilon_1 = \epsilon_2 = \infty$ the AT model reduces to a nearest-neighbour Ising model and is exactly soluble. [Γ now traces along the axis from (0, 0, 0) to (0, 0, 1).] Hence σ intersects the axes of the ω-space only at $(0, 0, r)$, $(0, r, 0)$ and $(r, 0, 0)$. Similar consideration for $\epsilon_1 = 0$, $\epsilon_2 = \epsilon_3$ shows that σ intersects the line $\omega_1 = 1$, $\omega_2 = \omega_3$ at $(1, r, r)$, etc.

Finally the $c_+ \leftrightarrow d_+$ symmetry of the staggered eight-vertex model (1) implies the invariance of σ under the transformation

$$\begin{aligned} \omega'_1 &= (1 + \omega_1 - \omega_2 - \omega_3)/(1 + \omega_1 + \omega_2 + \omega_3) \\ \omega'_2 &= (1 - \omega_1 + \omega_2 - \omega_3)/(1 + \omega_1 + \omega_2 + \omega_3) \\ \omega'_3 &= (1 - \omega_1 - \omega_2 + \omega_3)/(1 + \omega_1 + \omega_2 + \omega_3). \end{aligned} \tag{5}$$

This is a 'reflection' symmetry about the plane (3), relating the two portions of σ separated by (3). Note that the points $(\omega_1, \omega_2, \omega_3)$ and $(\omega'_1, \omega'_2, \omega'_3)$ are colinear with $(-1, -1, -1)$.

With this information, the geometry of the critical surface σ can be pictured as three bowl-shaped pieces sewn together at the line segments L_1 as shown schematically in figure 1. We conjecture that this is the case. Note that σ is known to pass through the light lines L and the heavy broken lines L_1 in figure 1. Now ω_1, ω_2 and ω_3 are monotonic along the thermodynamic path Γ; also Γ lies within one of the regions $\omega_i \geqslant \omega_j \geqslant \omega_k$. Assuming a regular-shaped σ surface as shown in figure 1, then Γ will in general intersect σ at two points, one on each side of the plane (3). Consequently, the general AT model has two phase transitions; the two transition temperatures are related by (5). A special situation arises when the two middle energies of the AT model are equal, ie $\epsilon_i = \epsilon_j \leqslant \epsilon_k$ with i, j, k distinct. In this case Γ lies in the plane $\omega_i = \omega_j$ and intersects σ at one of the lines L_1. Thus Γ and σ intersect only once and the AT model has only one transition. The critical point in this degenerate case is given by (3). This includes the case $\epsilon_1 = \epsilon_2 = \epsilon_3 > 0$ considered by Kramers and Wannier (1941). The conclusions are unchanged when one of the ω's vanishes, say $\epsilon_1 = \infty$. [Γ now lies in the $\omega_1 = 0$ plane and traces from (0, 0, 0) to (0, 1, 1).]

Our conjecture implies that both the staggered eight-vertex model and the Ising equivalent of the AT model have two phase transitions. This situation can also be seen from the trajectory of the thermodynamic paths Γ of these two models. In the Ising model, Γ again traces from (0, 0, 0) to (1, 1, 1), whereas in the staggered eight-vertex model a little reflection shows that Γ can be taken to trace from (1, 0, 0) to (0, 1, 0) in figure 1. In either case Γ intersects σ twice. A unique transition occurs only when Γ intersects σ at L_1. The condition for this to happen is $|J_i| = |J_j| \geqslant |J_k|$ for the Ising model and $c_+ = d_+$ for the staggered eight-vertex model. The former condition has previously been given by Wegner (1972) from symmetry considerations.

We point out in conclusion that the continuity assumption should always be taken with caution. While there exist other vertex models for which it is known to be invalid

(Wu 1972, 1974), the assumption seems plausible in the present model and is certainly consistent with all the known results. Indeed, from our analysis it would be most suprising if the general AT model does *not* exhibit two phase transitions.

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