Some rigorous results for the vertex model in statistical mechanics

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It is shown that the free and periodic boundary conditions are completely equivalent for the ice-rule (six-vertex) models in zero field. With an external direct or staggered field, we establish that in an ice-rule model the free and periodic boundary conditions are equivalent, and also equal to some special boundary conditions, either at sufficiently low temperatures or with sufficiently high fields in the appropriate direction. Regions of constant direct polarization are found. We also establish the existence of the spontaneous staggered polarization in an antiferroelectric using the Peierls argument.

I. INTRODUCTION

We consider in this paper some outstanding unsolved problems in the vertex models in statistical mechanics.[1] One problem whose solution has proven to be elusive in the past is the proof of the equivalence of the free and periodic boundary conditions for the six-vertex (ice-rule) models. In the solution of the ice-rule models obtained by Lieb,[2] it is crucial that a periodic boundary condition is used. Attempts in trying to show that the free boundary condition would yield the same solution have not succeeded.[3] Another problem which has not been considered before is the proof of the existence of a long-range order in an antiferroelectric. Such a proof is useful and desirable, especially since the only available calculation is that of the F model and is based on assumptions that appear to be difficult to justify.[4] In this paper we consider both of these problems.

In Sec. II we define the various vertex models and standardize the notation to be used in the ensuing proofs. The equivalence of various boundary conditions for the six-vertex models is considered in Sec. III using the weak-graph expansions. In particular, it is established that with no external field the free and periodic boundary conditions are completely equivalent. With an external direct and staggered field, we establish that the free and periodic boundary conditions are equivalent, and also equal to some special boundary conditions consistent with the energetically favored configurations, either at sufficiently low temperatures or with sufficiently high fields. These results, while expected, have not previously been proved. In Sec. IV, using the Peierls argument, we establish the existence of the spontaneous staggered polarization in a general antiferroelectric, including the F model. Several challenging unsolved problems related to the F model are presented in Sec. V.

II. BASIC DEFINITIONS

Let L be a two-dimensional square lattice of $N = n \times n$ vertices, with n even. The lattice edges of L which terminate in two vertices of L are called interior edges and those terminating in one vertex called exterior. A vertex which is the terminus of one or more exterior edges is called a boundary vertex and the set of such vertices is called the boundary of L. The vertex model is defined by placing arrows on the edges of L.

We follow the notation of Ref. 1 in defining the vertex models. The most general vertex model that can be defined on L is the 16-vertex model. The 16 arrow configurations that can occur at a vertex are shown in Fig. 1.

Let e_ξ be the energy associated with the vertex of type $\xi(=1,2,\ldots,16)$ and $\xi(i)$ be the type of configuration of the ith vertex. The partition function of the 16-vertex model is

$$Z = \sum \prod_{i=1}^{N} \omega_{\xi(i)} \tag{1}$$

where $\omega_\xi = \exp(-e_\xi/kT)$ is the Boltzman factor and the summation is extended to all arrow (or bond) configurations subject to a given boundary condition.

If all 16 ω's are nonzero, we have a 16-vertex problem. Otherwise we have an eight-vertex model if ω_ξ is nonzero for $\xi = 1, 2, \ldots, 8$ only, and a six-vertex problem if $\omega_\xi \neq 0$ for $\xi = 1, 2, \ldots, 6$. Some physical properties of the six-vertex models appear to be rather different from the corresponding eight- or 16-vertex ones, and it is the main purpose of this paper to stress and discuss them.

We study a general six-vertex model defined by[1]

$$\begin{aligned} &e_1 = \epsilon_1 - (h+v), \quad e_2 = \epsilon_1 + h + v, \\ &e_3 = \epsilon_2 - h + v, \quad e_4 = \epsilon_2 + h - v, \\ &e_5 = \epsilon_3 + s, \quad e_6 = \epsilon_3 - s \quad \text{for vertices on} \\ &\text{sublattice } A, \end{aligned} \tag{2}$$

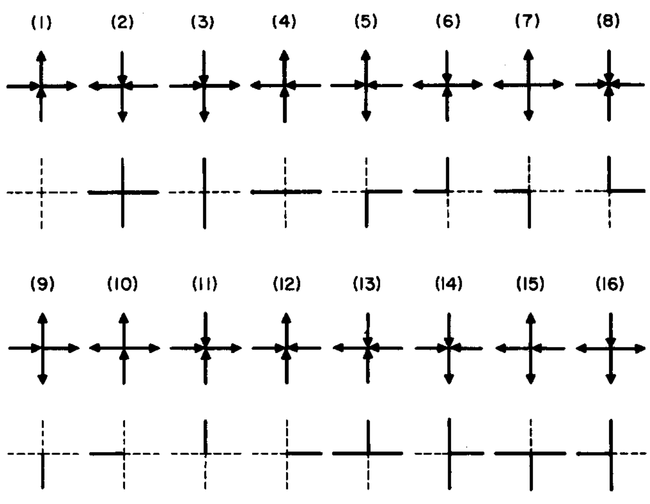

FIG. 1. The vertex configurations of the 16-vertex model and the associated bond configurations using the convention C_2.

$e_5 = \epsilon_3 - s, \quad e_6 = \epsilon_3 + s$ for vertices on sublattice B,

other $e_\xi = \infty$.

The electric fields included are the horizontal and vertical direct fields h and v, and the staggered quadrupole field s.

The general KDF model is defined by

$$\epsilon_2 > \epsilon_1, \quad \epsilon_3 > \epsilon_1. \tag{3}$$

In the absence of any field, the ground state configurations are X_1 and X_2, which correspond to complete filling of the lattice with vertices $\xi = 1, 2$, respectively. An equivalent definition is $\epsilon_1 > \epsilon_2$, $\epsilon_3 > \epsilon_2$, with ground states X_3 and X_4.

The general F model is defined by

$$\epsilon_1 > \epsilon_3, \quad \epsilon_2 > \epsilon_3. \tag{4}$$

For sufficiently small direct fields and without staggered field, the ground state configurations are X_{56} and X_{65}, where X_{56} is given by

$\xi(i) = 5, \quad i$ on sublattice B,

$\xi(i) = 6, \quad i$ on sublattice A.

X_{65} is obtained from X_{56} by interchanging 5 and 6. The same ground states are found for the generalization of the F model to the 16-vertex antiferroelectric

$$e_5 = e_6 = 0, \quad e_\xi > 0 \quad \text{for all } \xi \neq 5, 6. \tag{5}$$

We shall study the dependence of the free energy

$$\mathfrak{F} = -kT \lim_{N\to\infty} (1/N) \ln Z \tag{6}$$

on the boundary conditions. A boundary condition (BC) on the finite lattice L is expressed by a restriction on the arrow directions on the exterior edges of L. We speak of free boundary conditions (FBC) if there is no restriction, of periodic boundary conditions (PBC) if the two sets of horizontal (resp. vertical) exterior edges have identical arrow configurations. If the directions of the arrows on the exterior edges are all specified, we have a special boundary condition (SBC). We shall use SBC S_α, $\alpha = 1, 2, 3, 4, 56, 65$; to denote the boundary condition fitting to the configuration X_α. In Figs. 2a, b, S_1 and S_{56} are pictured. The boundary condition under which $\mathfrak{F}$ is defined will be denoted by the subscript F, P, or α.

As a consequence of the ice rule, the SBC S_α, $\alpha = 1, \ldots, 4$, completely determines the configuration X_α in the interior of L. Therefore, if the free energy (6) satisfies

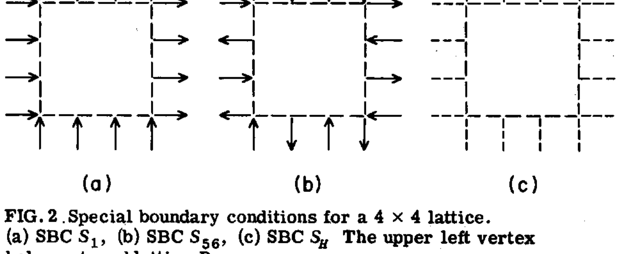

FIG. 2. Special boundary conditions for a 4×4 lattice. (a) SBC S_1, (b) SBC S_{56}, (c) SBC S_H The upper left vertex belongs to sublattice B.

$$\mathfrak{F}_F = \mathfrak{F}_\alpha$$

for $\alpha = 1, \ldots, 4$, then the system is in the frozen state X_α; in particular, the direct polarization is saturated in the corresponding direction.

Furthermore, the SBC S_{56} and S_{65}, which allow equal numbers of in and out arrows along the exterior rows and columns, only permit configurations in L with total direction polarization zero. Indeed, by the ice rule the vertical polarization is conserved from row to row, the horizontal from column to column. Therefore, if

$$\mathfrak{F}_F = \mathfrak{F}_{56} \text{ or } \mathfrak{F}_{65},$$

in some region in (h, v), the direct polarization is zero in that region.

Since the free energy is different for the different frozen states, we already see that the free energy cannot be independent of the BC. This difficulty disappears as soon as the weights satisfy

$$\omega_1, \ldots, \omega_8 > 0, \quad \omega_9, \ldots, \omega_{16} \geq 0. \tag{7}$$

In that case, a volume $n \times n$ with any BC can be imbedded in a volume $(n+2) \times (n+2)$ with any other BC, the energy difference being of the order of n. This implies then that the limiting free energy (6) does not depend on the BC (see, e.g., Ref. 5, Lemma 2.2.1).

It is useful to introduce representations of the vertex configurations in terms of bond graphs. The bond graphs in Fig. 1 are obtained by drawing bonds for each ← and ↓ arrow. Under this convention, which we shall call C_2, vertex 2 becomes the one with four bonds, configuration X_2 becomes X_B with bonds everywhere, and SBC S_2 becomes SBC S_B with bonds on all exterior edges. Analogously, we have the convention C_α, $\alpha = 1, \ldots, 4$, 56, 65, which carries X_α over into X_B and S_α into S_B.[6] In the bond language we also meet SBC S_H, Fig. 1c, in which no bonds (holes) appear on all exterior edges. In Sec. III the bond representations are used, together with the weak graph transformation, to prove the equivalence of boundary conditions in various cases.

In Sec. IV we use a representation of the arrow configurations in terms of closed polygons. For the 16-vertex antiferroelectric (5), we prove the existence of the spontaneous staggered polarization

$$P_0 = -\left(\frac{\partial \mathfrak{F}}{\partial s}\right)_{s=0+} \tag{8}$$

for sufficiently low temperatures. For the F model we also find equivalence of FBC to SBC S_{56} or S_{65}, which gives regions, where the direct polarization is zero and the staggered polarization is independent of the direct field.

III. EQUIVALENCE OF BOUNDARY CONDITIONS FOR THE ICE-RULE MODELS

Our proof is based on the application of the weak-graph expansion.[7] It was first shown by Nagle[8] that under the weak-graph expansion the six-vertex KDP and F models are transformed into eight-vertex models. For a general lattice model the weak-graph expansion will lead to a 16-vertex model. If all the transformed vertex weights are positive, one can then establish the equivalence of the boundary conditions as indicated above. The crux of matter is therefore to find a weak-graph expansion which will generate positive weights.

It will be convenient for our discussions to first review briefly the formulation of the weak-graph expansion. For simplicity we shall use bond graphs. For each edge connecting vertices i and j we introduce a matching factor

$$\tfrac{1}{2}[1 + c_{ij}(\xi_i)c_{ij}(\xi_j)] = 1, \quad \text{if } \xi_i \text{ and } \xi_j \text{ are compatible,} \qquad = 0, \quad \text{otherwise,} \tag{9}$$

where

$$c_{ij}(\xi_i) = c_{ji}(\xi_i) = 1, \quad \text{if } \xi_i \text{ has a bond on edge } ij, \qquad = -1, \quad \text{otherwise.} \tag{10}$$

Then the partition function takes the form

$$Z = \sum_{\xi_i=1}^{16} \prod_{\substack{\text{matching}\\ \text{edges}}} \tfrac{1}{2}[1 + c_{ij}(\xi_i)c_{ij}(\xi_j)] \prod_{i=1}^{N} \omega(\xi_i). \tag{11}$$

For FBC there are $2n(n-1)$ matching (interior) edges and for PBC there are $2n^2$, and for SBC S_B, $2n(n+1)$. Note that the matching factors for the exterior edges in the case of S_B have the form $\frac{1}{2}[1 + c_{ij}(\xi_i)]$.

Next we expand the product of the edge factors. Each term in the expansion is now a product of many $c_{ij}(\xi_i)c_{ij}(\xi_j)$ factors which can be conveniently represented graphically by drawing bonds between the connected vertices i and j. After rearrangement, (11) can be rewritten as

$$Z = c \sum_G \prod_{i=1}^{N} \left(\tfrac{1}{4} \sum_{\xi_i=1}^{16} \omega(\xi_i) \prod_{\substack{k\\ ik \text{ in } G}} c_{ik}(\xi_i) \right), \tag{12}$$

where the summation is extended to all bond graphs G, the constant $c = 1$ for PBC, $c = 2^{2n}$ for FBC and $c = 2^{-2n}$ for SBC S_B.

The summation inside the square bracket in (12) can be considered to be some new vertex weights ω'_ξ defined by the bond graphs G. Since the bonds in G are drawn only on the matched edges, which include the exterior edges for S_B and do not include them for FBC, we find

$$Z_P(\omega) = Z_P(\omega'), \qquad Z_F(\omega) = 2^{2n} Z_H(\omega'),$$
$$Z_B(\omega) = 2^{-2n} Z_F(\omega'). \tag{13}$$

The new weights are

$$\omega'(\xi_i) = \tfrac{1}{4} \sum_{\eta_i} \omega(\eta_i) \prod_{\substack{k\\ ik \text{ in } \xi_i}} c_{ik}(\eta_i). \tag{14}$$

Detailed expression of this linear transformation can be found in Eq. (408) of Ref. 1 and will not be reproduced here.

If the transformed weights ω'_ξ satisfy (7), then the thermodynamic limit (6) is independent of the BC. For the original weights ω_ξ we have then, by (13),

$$\mathfrak{F}_P(\omega) = \mathfrak{F}_F(\omega) = \mathfrak{F}_B(\omega). \tag{15}$$

Let us apply this basic idea to various six-vertex models.

(i) General six-vertex model in zero field: Here we show the equivalence $\mathfrak{F}_P = \mathfrak{F}_F$ for the general six-vertex model in zero field. Let $u_i = \exp(-\epsilon_i/kT)$, $i = 1, 2, 3$. Further, we take the fields $h = v = s = 0$ in (2). With convention C_2, the weak-graph expansion leads to the eight-vertex problem

$$\omega'_1 = \omega'_2 = \tfrac{1}{2}(u_1 + u_2 + u_3), \qquad \omega'_3 = \omega'_4 = \tfrac{1}{2}(u_1 + u_2 - u_3),$$
$$\omega'_5 = \omega'_6 = \tfrac{1}{2}(u_1 - u_2 + u_3), \qquad \omega'_7 = \omega'_8 = \tfrac{1}{2}(u_1 - u_2 - u_3),$$
$$\text{other } \omega'_\xi = 0. \tag{16}$$

Some of these weights may be negative, but if n is even the partition functions for PBC and SBC S_H are invariant under the reversal of sign of these weights.[9] Then we conclude from (13) that

$$\mathfrak{F}_P(\omega) = \mathfrak{F}_F(\omega)$$

at all temperatures. Strictly speaking, the above reasoning does not hold for the special temperature, where two of the transformed weights are zero; but there the result follows by continuity of the free energies in the temperature. The equality of $\mathfrak{F}_P$ and $\mathfrak{F}_F$ to $\mathfrak{F}_B$ does not follow generally, because the FBC partition function of the transformed weights is not invariant under the reversal of sign. For $T < T_c$ we also establish in the following that $\mathfrak{F}_P(\omega) = \mathfrak{F}_F(\omega) = \mathfrak{F}_\alpha(\omega)$, where X_α is the ground state configuration.

(ii) General six-vertex model in direct and staggered fields: For the general six-vertex model in nonzero fields, we now show that $\mathfrak{F}_F = \mathfrak{F}_P = \mathfrak{F}_\alpha$, $\alpha = 1, 2, 3, 4, 56, 65$, provided that the direct or the staggered field is large enough in the appropriate direction.

Consider the general six-vertex model (2). We put $H = h/kT$, $V = v/kT$, $t = s/kT$. With convention C_2, the weak-graph expansion leads to the weights

$$\omega'_1, \ldots, \omega'_8 = \tfrac{1}{2}[u_1 \cosh(H + V) \pm u_2 \cosh(H - V) \pm u_3 \cosh t],$$
$$\omega'_9, \ldots, \omega'_{16} = \tfrac{1}{2}[-u_1 \sinh(H + V) \pm u_2 \sinh(H - V) \pm u_3 \sinh t], \tag{17}$$

each combination of the signs occurring twice. The transformed weights ω' satisfy the positivity condition (7) if

$$u_1 \cosh(H + V) > u_2 \cosh(H - V) + u_3 \cosh t,$$
$$-u_1 \sinh(H + V) \geq u_2 \sinh|H - V| + u_3 \sinh|t|. \tag{18}$$

This may always be satisfied at a given temperature by taking $h \approx v \ll -|s|$, independent of the other parameters. So then we have

$$\mathfrak{F}_F(\omega) = \mathfrak{F}_P(\omega) = \mathfrak{F}_2(\omega). \tag{19}$$

By using the convention S_α, $\alpha = 1, \ldots, 4$, we find in the same way that

$$\mathfrak{F}_F(\omega) = \mathfrak{F}_P(\omega) = \mathfrak{F}_\alpha(\omega) \tag{20}$$

if the direct field is large enough in the appropriate direction. The system is then in the frozen state X_α with saturated direct polarization. Similarly if the vertex (2) is favored, i.e., $e_2 < e_\alpha$, $\alpha \neq 2$, then (18), and hence (19), are always satisfied at sufficiently low temperatures. The same conclusion holds for (20) from which we conclude that if the vertex energies are such that the vertex (α), $\alpha = 1, \ldots, 4$, is favored, the system is in the frozen state X_α at sufficiently low temperatures, a result borne out by the exact solution.[1]

In the case that the configuration 56 is favored, we use the convention C_{56} so that the weak-graph transformation gives the weights

$$\omega_1', \ldots, \omega_8' = \tfrac{1}{2}[u_3 \cosh t \pm u_1 \cosh(H+V) \pm u_2 \cosh(H-V)],$$

$$\omega_9', \ldots, \omega_{16}' = \tfrac{1}{2}[u_3 \sinh t \pm u_1 \sinh(H+V) \pm u_2 \sinh(H-V)]. \quad (21)$$

Using (7), this leads to the conditions

$$u_3 \cosh t > u_1 \cosh(H+V) + u_2 \cosh(H-V),$$
$$u_3 \sinh t \geq u_1 \sinh|H+V| + u_2 \sinh|H-V|. \quad (22)$$

By taking the staggered field $s \gg |h| \approx |v|$, at a given temperature, this can always be satisfied. Then

$$\mathfrak{F}_F(\omega) = \mathfrak{F}_P(\omega) = \mathfrak{F}_{56}(\omega). \quad (23)$$

With convention C_{65}, we find that

$$\mathfrak{F}_F(\omega) = \mathfrak{F}_P(\omega) = \mathfrak{F}_{65}(\omega) \quad (24)$$

if $s \ll -|h| \approx -|v|$. In both cases, the direct polarization is zero. Similarly if the vertex energies are such that the configuration 56 is favored and $s > 0$, then (22), hence (23), is always satisfied at sufficiently low temperatures. Likewise, we find (24) to hold at sufficiently low temperatures and $s < 0$ if the configuration 65 is favored. The proof breaks down for $s = 0$ and h or $v \neq 0$ (the F model in a direct field). Fortunately an alternate proof, which is valid for the F model with *arbitrary s*, exists. For continuity or reading, details of the latter proof will be given in the Appendix. We remark that, under (23), (24), or (A1), the free energy is independent of the direct field h and v. It follows that, in particular from (A1), P_0 *is independent of h and v.*

(iii) The KDP and F models: While the results in (ii) above are sufficiently general, it is illuminating to specialize these conclusions to some special cases:

KDP model in zero field: The exact transition temperature T_c is known to be given by $u_1 = u_2 + u_3$; hence the condition (18) corresponds to $T < T_c$. We conclude that the system is in a frozen state for $T < T_c$, a result known from the exact solution.[1] More generally we have

$$\mathfrak{F}_F(\omega) = \mathfrak{F}_P(\omega) = \mathfrak{F}_\alpha(\omega), \; T < T_c \quad (25)$$

where X_α, $\alpha = 1, 2, 3, 4$, is the ground state configuration.

KDP model in a vertical field:—The condition (18) for the system to be in a frozen state X_2 becomes

$$u_1 \cosh V > u_2 \cosh V + u_3, \quad v \leq 0. \quad (26)$$

It can be easily checked that the temperature given by (26) is lower than the exact transition temperature T_c [cf. (327) in Ref. 1]. Consequently the conclusion derived here is somewhat weaker than the known exact result[1] which states that the system is frozen for $T < T_c$.

F model in a zero field: we conclude that

$$\mathfrak{F}_F(\omega) = \mathfrak{F}_P(\omega) = \mathfrak{F}_{56}(\omega) = \mathfrak{F}_{65}(\omega), \quad T < T_c, \quad (27)$$

where the transition temperature T_c is given by the relation $u_3 = u_1 + u_2$.[1]

F model in a staggered field: For $s > 0$ we find from (22) that (23) holds if

$$u_3 \cosh t > u_1 + u_2. \quad (28)$$

We conjecture that (28) yields a lower bound on the exact transition temperature for a nonzero staggered field. If also a direct field v is present, Eqs. (18), (22), and (A2) determine regions of constant polarization 0, ± 1 in the (v, T) plane, which have the same qualitative shape as the regions found for $s = 0$ (Ref. 1, Fig. 31).

IV. EXISTENCE OF P_0

In this Section we use Peierls argument[10] to establish the existence of the spontaneous staggered polarization P_0 in the general antiferroelectric (5).[11] We need only to show the existence for the SBC S_{56}. For the eight- and 16-vertex models the result is independent of the BC. For the F model, we then use the equivalence of SBC S_{56} with other BC established in Sec. III in conjunction with the standard concavity argument[12] to extend the existence of P_0 to FBC and PBC.

The first step of our proof is to introduce for a vertex model a graph representation consisting of closed contours. Consider a lattice L' composed of all lattice points of L and also the intersecting points of the diagonals in L. The edges of L' are the half-diagonals in L. An example of L' and its relationship with L is shown in Fig. 3 for a 4×4 lattice. Consider the four quadrants belonging to a vertex of L. If a quadrant is bounded by two arrows pointing in or by two arrows pointing out, we draw a bond along the half-diagonal bisecting the quadrant. Then one can easily see that there are always an even number of bonds meeting at any interior vertex of L'. This is also the case on the boundary of L', as illustrated in Fig. 3, if one assumes SBC S_{56} or S_{65} in L. The result is then a *one-to-one correspondence between the arrow configurations on L* (assuming S_{56} or S_{65}) *and the closed-polygonal configuration on L'*.[13] This correspondence holds even if some of the vertex energies of L are $+\infty$ such as in the F model. This only restricts the allowed contours on L' and will not affect the following discussions.

With these preliminaries, it is now relatively easy to formulate the Peierls argument. Since our proof follows closely the standard argument,[12] we shall only point out the essential points. From (8) we have

$$P_0 = \lim_{n\to\infty} (1/N)\langle n_+ - n_-\rangle,$$
$$n_+ = n_{5B} + n_{6A}, \quad n_- = n_{5A} + n_{6B}, \quad (29)$$

where n_{5B} denotes the number of (5) vertices on sublattice B, etc., and $\langle\,\rangle$ is the thermal average. Denote the vertices of n_+ by "+" and n_- by "—", as shown in Fig. 3, then the contours on L' separate seas of "+" from "seas" of "—". In fact, the contours go through all vertices except (5) and (6). Furthermore, the boundary of L' cannot be "—". Let

$$\epsilon = \min\{e_1, \ldots, e_4, \tfrac{1}{2}e_7, \tfrac{1}{2}e_8, e_9, \ldots, e_{16}\} > 0. \quad (30)$$

Then each bond segment on L' has a weight of at most $\exp(-\epsilon/2kT)$. The standard argument then gives

$$(1/N)\langle n_-\rangle \leq \sum_{b\geq 4} (b/4)^2\, 3^{b-1} e^{-b\epsilon/2kT}. \quad (31)$$

Similarly the average fraction of vertices other than (5) and (6) is bounded by

$$1 - (1/N)\langle n_+ + n_-\rangle \leq \sum_{b \geq 4} 3^{b-1} e^{-b\epsilon/2kT}. \quad (32)$$

The right-hand sides of (31) and (32) can be made arbitrarily small at sufficiently low temperatures. It follows then $(1/N)\langle n_+ - n_-\rangle$ is nonzero and hence $P_0 > 0$ exists. This completes the proof.

V. DISCUSSIONS

Our methods are especially suited for discussion of the low temperature properties of the vertex models. They do not allow us to establish the equivalence of FBC and PBC for the six-vertex models at high temperature, except in the case of zero field where the existence of extra symmetries permits us to complete the proof.

We wish to point out that it is a mere consequence of the ice rule that the six-vertex model can be put in a frozen state X_α, $\alpha = 1, 2, 3, 4$, if the vertex α is favored. This frozen state disappears as soon as the ice rule is violated. Indeed, another way to state these properties is to say that the free energy is not a strictly convex function of the direct fields h and v at sufficiently low temperatures. However, it has been shown[14] that the eight- or the 16-vertex model in a direct field h, v and a staggered field is equivalent to an Ising model on a square lattice with finite, but short-ranged, interactions. Recently, Griffiths and Ruelle[15] have shown that the free energy of such lattice system is a strict convex function of the translationally invariant interactions. Since the translationally invariant interactions of the corresponding Ising model depend linearly on h and v, the free energy of the eight- or 16-vertex models will be a strictly convex function of h and v. This then rules out the occurrence of the frozen states in these models.

To conclude our discussion, we list some unsolved problems related to the F model which appear to be particularly challenging:

(i) Are $\mathfrak{F}_F$ and $\mathfrak{F}_P$ analytic in s near $s = 0$ at sufficiently high temperatures? We note that Baxter[16] has shown $\mathfrak{F}_P$ is singular at $s = 0$ when $T = 2T_c$.

(ii) What is the decay of the correlation functions? Is it exponential like in an ordinary lattice gas with short-ranged interactions?

(iii) We expect $P_0 = 0$ for $T > T_c$ and a rigorous proof of this fact appears to be lacking.

(iv) In the F model, like in an antiferromagnet, the translational invariance is broken at low temperatures and under small direct fields. In the (v, T) plane, for example, we then expect a boundary B_1 which separates the regions where $P_0 = 0$ and $P_0 \neq 0$, as in an ordinary antiferromagnet. Does this boundary coincide with the boundary B_2 separating the region of vertical polarization $y = 0$ and $y \neq 0$? Note that B_1, but not B_2, disappears if there is a nonzero staggered field. On the other hand, as discussed in the above, it is B_2, not B_1, which disappears when one goes over to an eight-vertex case.

(v) Does the staggered susceptibility diverge along the boundary B_1 separating regions of different long-range orders. Here we note that Baxter found that the staggered susceptibility diverges at $T = 2T_c$!

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APPENDIX

In this appendix we prove the following result for the F model specified by (2) and (4):

$$\mathfrak{F}_F(s) = \mathfrak{F}_P(s) = \mathfrak{F}_{56}(s), \quad s \geq 0,$$
$$= \mathfrak{F}_{65}(s), \quad s \leq 0, \tag{A1}$$

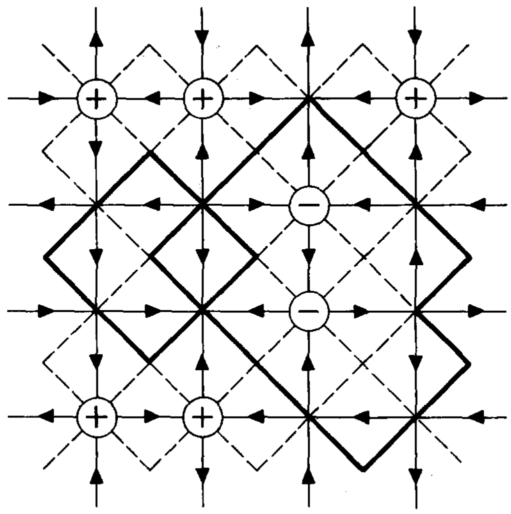

FIG. 3. Relationship between the lattices L and L'. Arrow configurations on L are mapped into closed-polygonal configurations on L'. The mapping is one-to-one if L assumes the SBC S_{56} as shown here. The polygons on L' separates the + and − vertices on L and the upper left vertex belongs to sublattice B.

provided that

$$T \leq \min\{T_c, (\epsilon + |s|)/\ln 3\}. \tag{A2}$$

Here T_c is the transition temperature of the F model in zero field, and ϵ is the minimum vertex energy defined by (29). First we establish the following lemma:

Lemma:

$$\left.\begin{array}{ll} Z_{56} \geq Z_{65}, & s \geq 0 \\ Z_{65} \geq Z_{56}, & s \leq 0 \end{array}\right\}, T \leq T_c.$$

Proof: The partition functions Z_{56} and Z_{65} are independent of h and v. Consider Z_{56} and Z_{65} as functions of the complex variable $z = \exp(s/kT)$. The boundary condition S_{56} can be simulated by taking $z = \infty$ at the boundary of the volume. It then follows from the work of Suzuki and Fisher[17] (see also Chang *et al.*[18]), that the partition function belongs to the Lee–Yang class (Ref. 17, Def. 1) for $T \leq T_c$, so that then we have[19]

$$Z_{56}(z) \neq 0, \quad |z| \geq 1. \tag{A3}$$

Also by arrow reversal we have, for all z,

$$Z_{56}(z) = Z_{65}(z^{-1}), \tag{A4}$$

which says that, for $|z| = 1$,

$$Z_{56}(z) = Z_{65}(z^*) = Z_{65}{}^*(z). \tag{A5}$$

Hence the function

$$f(z) \equiv Z_{65}(z)/Z_{56}(z)$$

is analytic in z for $|z| \geq 1$ ($z = \infty$ included) and satisfies

$$|f(z)| = 1 \quad \text{for } |z| = 1. \tag{A6}$$

The maximum modulus principle then gives

$$|f(z)| \leq 1 \quad \text{for all } |z| \geq 1. \tag{A7}$$

In particular this leads to the first part of the Lemma. The second part is obtained by symmetry.

Now we proceed to prove (A1). First consider $s \geq 0$. Using the notation of Sec. IV, we note that with FBC on L the contours can terminate at the boundary of L'. So any configuration can be described in terms of connected paths λ_k beginning and ending at the boundary together with closed contours. The paths λ_k divide L into subvolumes Λ_j which have the BC S_{56} or S_{65} (adjacent subvolumes have opposite BC). Since the polygon-to-arrow-configurations correspondence is two-to-one, then to each path $\{\lambda_k\}$ on L' there correspond two terms of the form

$$\prod_k e^{-u(\lambda_k)/kT} \prod_j Z_\alpha(\Lambda_j) \tag{A8}$$

in the partition function $Z_F(L)$. Here α stands for either 56 or 65 and $u(\lambda_k)$ is the sum of the vertex energies along the path λ_k. Thus, by the lemma just proved, an upper bound on $Z_F(L)$ results for $s \geq 0$ if we replace all Z_α in (A8) by Z_{56}. Furthermore, we note that

$$Z_{56}(L) \geq \prod_k e^{s|\lambda_k|/kT} \prod_j Z_{56}(\Lambda_j) \tag{A9}$$

since every term on the right-hand side of (A9) appears in $Z_{56}(L)$ but not vice versa; here $|\lambda_k|$ is the number of vertex points along λ_k. Using (29), we have also

$$u(\lambda_k) \geq \epsilon |\lambda_k|. \tag{A10}$$

Combining (A8)–(A10) and replacing Z_α by Z_{56} in (A8), we then obtain the bound

$$Z_F \leq Z_{56} D, \qquad s \geq 0, \tag{A11}$$

where

$$D = \sum_{\{\lambda_k\}} e^{-(\epsilon+s)|\lambda_k|/kT}. \tag{A12}$$

Now each λ_k must pass through at least one point on the boundary, we obtain

$$D \leq \sum_{\alpha=0}^{4n+4} \binom{4n+4}{\alpha} \left(\sum_{b\geq 1} 3^b e^{-b(\epsilon+s)/kT} \right)^\alpha = \left(1 + \sum_{b \geq 1} 3^b e^{-b(\epsilon+s)/kT}\right)^{4n+4}, \tag{A13}$$

where $4n+4$ is the number of vertices on the boundary of L'. The series in (A13) converges if

$$kT < (\epsilon + s)/\ln 3. \tag{A14}$$

On the other hand, we have generally[1]

$$Z_{56} \leq Z_P \leq Z_F. \tag{A15}$$

Thus we deduce the first statement ($s \geq 0$) of (A1) on taking the thermodynamic limits of (A11) and (A15). The statement with $s \leq 0$ can be similarly obtained or by symmetry considerations.

We remark that the result (A1) is valid for arbitrary s. Furthermore, for small $|s|$, the temperature bound (A2) is better than the one obtained in Sec. III using the weak-graph expansions.

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[1]E. H. Lieb and F. Y. Wu, in *Phase Transitions and Critical Phenomena*, Vol. I, edited by C. Domb and M. S. Green (Academic, London, 1972).

[2]E. H. Lieb, Phys. Rev. Letters 18, 692, 1046 (1967); 19, 108 (1967).

[3]See discussions in Ref. 1.

[4]R. J. Baxter, J. Phys., C, L94 (1973) and private communications.

[5]D. Ruelle, *Statistical Mechanics* (Benjamin, New York, 1969).

[6]The mapping C_{56} is that given in Fig. 5 of Ref. 1.

[7]J. F. Nagle, J. Math. Phys. 9, 1007 (1968).

[8]J. F. Nagle, J. Math. Phys. 7, 1492 (1966).

[9]C. Fan and F. Y. Wu, Phys. Rev. B 2, 723 (1970).

[10]R. Peierls, Proc. Cambr. Phil. Soc. 32, 477 (1936); R. B. Griffiths, Phys. Rev. 136, A437 (1964); R. L. Dobrushin, Teor. Veroyat. Primen. 10, 209 (1965) [Theory Prob. Appl. 10, 193 (1965)]; Funkt. Anal. Pril. 2, 44 (1968) [Funct. Anal. Appl. 2, 302 (1968)].

[11]If the vertex energies satisfy certain symmetry such as in an eight-vertex model in zero field, this result then also establishes the existence of a spontaneous polarization of a ferroelectric in a direct field in 45° direction in the first quadrant. See discussions in H. J. Brascamp, H. Kunz, and F. Y. Wu, J. Phys. C. L164 (1973).

[12]See, for example, discussion by R. B. Griffiths, in Ref. 1.

[13]The correspondence is two-to-one for periodic lattices.

[14]Pp. 348-53, Ref. 1.

[15]R. B. Griffiths and D. Ruelle, Comm. Math. Phys. 23, 169 (1971).

[16]R. J. Baxter, Phys. Rev. B 1, 2199 (1970).

[17]M. Suzuki and M. E. Fisher, J. Math. Phys. 12, 235 (1970).

[18]K. S. Chang, S. Y. Wang, and F. Y. Wu, Phys. Rev. A 4, 2324 (1971).

[19]The conditions of Theorem 1 in Ref. 17 imply that $Z(z)$ here is of the *Lee–Yang class*.